Complex Variables

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Contents

1 Complex Numbers

1.1 Definitions

Definition 1.1.1 (Complex Number)**.** A *complex number* is a point $z = (x, y)$ of the plane, with $x, y \in \mathbb{R}$. We denote z by

$$z = x + iy.$$

The set of complex numbers is denoted by $\mathbb{C}$.

Definition 1.1.2 (Real and Imaginary Parts)**.** Let $z = x + iy$ be a complex number. We define the following:

(1) $\mathfrak{Re}(z) = x$ is called the *real part* of z.

(2) $\mathfrak{Im}(z) = y$ is called the *imaginary part* of z.

Now that we have defined complex numbers to be points on the plane, we now need to define addition and multiplication over the complex numbers in order to make it a ring.

Definition 1.1.3. Let $z_1 = x_1 + iy_1$ and let $z_2 = x_2 + iy_2$ be complex numbers. Then the sum $z_1 + z_2$ is defined by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

The product $z_1 z_2$ is defined by

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2).$$

Remark. The definition of the sum is consistent with the notation $x + iy$; for instance, $(2) + (3i) = (2 + 0i) + (0 + 3i) = 2 + 3i$. Notice how the definition of the $+$ sign changes.

Moreover, the way we defined the product is important. If you work it out, we can see that

$$i^2 = i \times i = (0 + i) \times (0 + i) = -1.$$

Thus, $i^2 = -1$ and so i is a solution to the equation $z^2 + 1 = 0$.

Definition 1.1.4. Let $z \in \mathbb{C}$. Then we define the quantity $-z$ to be

$$-z = (-x) + i(-y).$$

Definition 1.1.5. We define $0 \in \mathbb{C}$ to be the quantity

$$0 := 0 + 0i.$$

Moreover, we define $1 \in \mathbb{C}$ to be

$$1 := 1 + 0i.$$

Definition 1.1.6. Let $z \in \mathbb{C}$, and $z = x + iy \neq 0$. Then we define z^{-1} to be

$$z^{-1} := \left(\frac{x}{x^2 + y^2}\right) + i\left(\frac{-y}{x^2 + y^2}\right).$$

To see where the equation for the inverse comes from, we can solve the linear system such that for $w = a + ib$, $zw = 1$. Multiplying them out, we get that

$$zw = 1 \iff (xa - yb) + i(ya + bx) = 1 + 0i \tag{1}$$

$$\iff \begin{cases} xa - yb = 1 \\ ya + bx = 0 \end{cases} \tag{2}$$

$$\iff \begin{cases} a = \frac{x}{x^2+y^2} \\ b = \frac{-y}{x^2+y^2} \end{cases} . \tag{3}$$

We know that we want $i^2 = -1$, and the usual rules of sums and products apply; in particular, we want the field axioms to hold. Thus

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + iy_1x_2 + x_1iy_2 + iy_1iy_2 \\ &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2). \end{aligned}$$

1.2 Basic Algebraic Properties

Recall the field axioms.

Proposition 1.2.1 ($\mathbb{C}$ is a field)**.** Let $z_1, z_2, z_3 \in \mathbb{C}$. Then the following holds:

(1) (Associativity.) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1z_2)z_3 = z_1(z_2z_3)$

(2) (Commutativity.) $z_1 + z_2 = z_2 + z_1$ and $z_1z_2 = z_2z_1$.

(3) (Distributivity.) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.

(4) (Identities.) We have that $z + 0 = z$ and $z \cdot 1 = z$ for all $z \in \mathbb{C}$.

(5) (Inverses.) For all $z \in \mathbb{C}$ and $z \neq 0$, then $z + (-z) = (-z + z) = 0$ and $zz^{-1} = z^{-1}z = 1 \in \mathbb{C}$.

Remark. Recall that all fields are integral domains. In other words, if $z_1z_2 = 0$, then at least one of z_1, z_2 are 0.

Example. Suppose we want to compute $(1+i)/(3+2i)$. A neat method is to multiply the numerator and the denominator by $(3 - 2i)/(3 - 2i) = 1$. Then we get

$$\frac{1+i}{3+2i} = \frac{1+i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{3-2i+3i-2i^2}{3^2-(2i)^2} = \frac{5+i}{9+4} = \frac{5+i}{13} = \frac{5}{13} + i\frac{1}{13}.$$

The crucial move here is that we were able to convert the denominator into a real number, thereby isolating the computations to the top. Since the formula for the inverse of a complex number is unwieldy, this is a very useful tool.

1.3 Vectors and Moduli

We have since defined our complex number as a point in the plane, characterized by two real numbers. This means that our complex numbers are precisely isomorphic to $\mathbb{R}^2$, given by the isomorphism

$$z = x + iy \mapsto (x, y) \in \mathbb{R}^2.$$

Moreover, since we can consider complex numbers to be points on a plane, it makes sence to give a notion of magnitude, or distance from the origin:

Definition 1.3.1 (Modulus, Magnitude)**.** The *modulus* or *magnitude* of a complex number $z = x+iy$ is

$$|z| := \sqrt{x^2 + y^2} = \sqrt{\Re(z)^2 + \Im(z)^2}.$$

1.4 Triangle Inequality

Theorem 1.4.1. Let $z_1, z_2 \in \mathbb{C}$. Then

(1) $|z_1 + z_2| \leq |z_1| + |z_2|$.

(2) $|z_1 + z_2| \geq ||z_1| - |z_2||$.

We have equality if and only if the angle between the vector representation of z_1 and z_2 are parallel.

[PROOF]

Proof. 1. proof

2. Foremost, we prove that $|z_1 + z_2| \geq |z_1| - |z_2|$. Note that

$$\begin{aligned} |z_2| &= |z_1 + z_2 - z_2| \\ &\leq |z_1 + z_2| + |-z_2| \\ &= |z_1 + z_2| + |z_2|. \end{aligned}$$

Thus $|z_1 + z_2| \geq |z_1| - |z_2|$. In the same way, we can also prove that $|z_1 + z_2| \geq |z_2| - |z_1|$. Therefore, $|z_1 + z_2| \geq ||z_1| - |z_2||$.

□

We can represent this theorem geometrically: [IMAGE?]

Corollary 1.4.2. Let $z_1, \dots, z_n \in \mathbb{C}$. Then

$$|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|.$$

Proof. The proof is done by induction. □

1.5 Complex Conjugation

Definition 1.5.1. Let $z = x + iy \in \mathbb{C}$. The *(complex) conjugate* of z, denoted $\overline{z}$, is defined as

$$\overline{z} := x - iy.$$

We can visually see how the two differ: [IMAGE?]
Immediately, the complex conjugate $\overline{z}$ is interesting:

Proposition 1.5.1. Let $z \in \mathbb{C}$. Then

(i) $\overline{\overline{z}} = z$

(ii) $|\overline{z}| = |z|$

(iii) $z\overline{z} = |z|^2$.

Proof.

(i) We have that $\overline{\overline{z}} = x - (-iy) = x = iy = z$.

(ii) $|\overline{z}| = |x + (-y)i| = \sqrt{x^2 + (-y^2)} = \sqrt{x^2 + y^2} = |z|$.

(iii)

$$\begin{aligned} z\overline{z} &= (x+iy)(x-iy) \\ &= x^2 - (iy)^2 \\ &= x^2 + y^2 \\ &= |z|^2. \end{aligned}$$

□

The last property of this proposition is the most interesting. It can be thought of as a counter rotation.

How can we use these formulas to do quick calculations on complex numbers without relying on the expansion of z as $x + iy$?

Lemma 1.5.2. Let $z \in \mathbb{C}$, and $z \neq 0$. Then

$$\begin{aligned} \Re\left(\frac{1}{z}\right) &= \frac{\Re(z)}{|z|^2} \\ \Im\left(\frac{1}{z}\right) &= \frac{-\Im(z)}{|z|^2}. \end{aligned}$$

Proof. In order to compute the real part, we note that

$$\Re\left(\frac{1}{z}\right) = \Re\left(\frac{\overline{z}}{z\overline{z}}\right) = \Re\left(\frac{\overline{z}}{|z|^2}\right) = \frac{1}{|z|^2}\Re\left(\overline{z}\right) = \frac{\Re(z)}{|z|^2}.$$

The proof for the second equation is similar, except $\Im(\overline{z}) = -\Im(z)$. □

Another useful proposition is how the conjugate behaves under the operations of addition and subtraction:

Proposition 1.5.3. Let $z_1, z_2 \in \mathbb{C}$. Then

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(ii) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

The following is important, since it allows us to interpret the function $\Re(z)$ as a linear combination of vectors in $\mathbb{C}$. This is why if we have a complex solution to a homogeneous differential equation, then we can simply take the real part and obtain a solution. which is physical

Proposition 1.5.4. Let $z \in \mathbb{C}$. Then

$$\begin{aligned} \Re(z) &= \frac{z + \overline{z}}{2} \\ \Im(z) &= \frac{z - \overline{z}}{2i} \end{aligned}$$

Proposition 1.5.5.

1. $|z_1 z_2| = |z_1| \cdot |z_2|$
2. If $z_2 \neq 0$, then $|z_1/z_2|$... [etc]

1.6 Exponential Form

So far we have thought of complex numbers in terms of Cartesian coordinates (x, y). In physical settings, it is sometimes more useful to think of them as polar coordinates, in terms of (r, θ).

For any $(x, y) \neq (0, 0)$ in the plane, there is a unique $\theta \in (-\pi, \pi]$ that works.

Definition 1.6.1 (Argument of a Complex Number)**.** Let $z = x + iy \in \mathbb{C}$ with $z \neq 0$. Then the *argument* of z is a real number θ such that

$$\begin{cases} x = |z|\cos\theta \\ y = |z|\sin\theta \end{cases}.$$

The set of arguments of z is denoted by $\arg(z)$.

The principal value of $\arg(z)$, denoted by $Arg(z)$ is the unique $\theta \in \arg(z) \cap (-\pi, \pi]$.

Claim (Euler's Formula)**.** Let $\theta \in \mathbb{R}$. Then we claim

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

[FINISH EXPONENTIAL STUFF]

1.7 Products & Powers

Proposition 1.7.1. Let $r_1, r_2 \geq 0$ and $\theta_1, \theta_2 \in \mathbb{R}$. Then

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

Proof. Proof by expanding into sins and cosines. □

Corollary 1.7.2. Let $r > 0$ and $\theta \in \mathbb{R}$. Then

$$(re^{i\theta})^{-1} = \frac{1}{r}e^{-i\theta}.$$

Proof. Multiplying; □

Theorem 1.7.3 (DeMoivre)**.** Let z be the complex number $z = r(\cos\theta + i\sin\theta)$. Then z^n is

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Proof. We can write z as $re^{i\theta}$. Thus,

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}.$$

[UNDUCTION] □

2 Holomorphic Functions

2.1 Functions and Mappings

Here, we are interested in functions whose inputs and outputs are not real variables but complex.

Definition 2.1.1 (Complex Function)**.** Let $S \subseteq \mathbb{C}$. A complex-valued function $f : S \to \mathbb{C}$ on S is a mapping from S to $\mathbb{C}$ such that each $z \in S$ is assigned a unique complex number $f(z)$, called the value of f at z or the image of z by f.

The set S is called the domain of f. The image (or range) of f is defined by

$$\operatorname{Im} f := \{f(z) : z \in S\}.$$

Example. One example of a complex function is $f(z) = z^2$, which maps complex inputs to complex outputs. The function $f(z) = 1/z$ is a function on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

However, $f(z)$ such that $z \mapsto \sqrt{z}$ is not well-defined, as there is not a unique value for the square root. However, $z \mapsto$ the principal square root is well-defined.

Remark. If we consider a complex number with the representation $z = x + iy$, we can decompose a complexly-valued function f into

$$f(z) = u(x, y) + iv(x, y)$$

where u and v are functions of two real varibles such that $u, v : \mathbb{R} \to \mathbb{R}$.

With these definitions in mind, it is worth identifying some commonly-seen transformations:

(1) Translation: $f(z) = z + z_0$ for a fixed complex number z_0.

(2) Scaling: $f(z) = rz$ where $r \in \mathbb{R}$. Note that if we allowed r to be complex, this would amount to a rotation.

(3) Rotation: $f(z) = e^{i\theta} z$ is the rotation centered at 0 with angle θ.

2.1.1 The Mapping $w = z^2$

Since we are in the complex plane, in order to map the input and output spaces, we would need to somehow have access to 4-Dimensional space.

2.2 Limits

In order to help with our definitions of limits in the complex plane, the following topological definition will be very useful.

Definition 2.2.1 (ε-neighborhood)**.** An ε-neighborhood of z_0 is an open disk of the form

$$B_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}.$$

for some $\varepsilon > 0 \in \mathbb{R}$.

We tend to use this definition when we specify behaviors that occur locall around a point in the complex plane (such as limits, etc.). Visually, this is a circle on the complex plane:

Definition 2.2.2 (Deleted ε-neighborhood)**.** A deleted ε-neighborhood of z_0 is the set $B_\varepsilon(z_0)$ with z_0 removed:

$$B_\varepsilon(z_0) \setminus \{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon\}.$$

The notion of deletion is useful in ignoring the behavior that may occur at some single point.

Definition 2.2.3 (Limit of a Complex Function)**.** Let $z_0 \in \mathbb{C}$. Let f be a function defined on a deleted ε-neighborhood of z_0. Then we say that

$$\lim_{z \to z_0} f(z) = w_0$$

if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any z such that $0 < |z - z_0| < \delta$, we have that $|f(z) - w_0| < \varepsilon$.

Intuitively, what this means is that as z approaches z_0, $f(z)$ approaches w_0. That is, there is a deleted δ-neighborood such that the behavior of $f(z)$ is contained within $B_\varepsilon(w_0)$.

Theorem 2.2.1. If the limit of f at a point z_0 exists, then it is unique.

Proof. Suppose there were two limits, w and w' such that $w \neq w'$. Then

$$\lim_{z \to z_0} f(z) = w$$

and

$$\lim_{z \to z_0} f(z) = w'.$$

Then let $\varepsilon = |w - w'|/2$. Then there is $\delta, \delta' > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w| < \varepsilon$$

and

$$0 < |z - z_0| < \delta' \Rightarrow |f(z) - w'| < \varepsilon$$

However, consider $\delta^* = \min(\delta, \delta')$. Then we have that

$$0 < |z - z_0| < \delta^* \Rightarrow |f(z) - w| < \varepsilon \text{ and } |f(z) - w'| < \varepsilon.$$

By the triangle inequality,

$$|w - w'| = |w - f(z) + f(z) - w'| \leq |f(z) - w| + |f(z) - w'| < 2\varepsilon = |w - w'|$$

which is impossible. Therefore $w = w'$. □

Definition 2.2.4. For any ε-neighborhood od w_0, there is a deleted δ-neighborhood of z_0 such that the image of $B_\delta(z_0)\{z_0\}$ is included in $B_\varepsilon w_0$.

2.3 Limit Theorems

Theorem 2.3.1. Let $z_0 = x_0 + iy_0$, and $w_0 = v_0 + iv_0$ be in $\mathbb{C}$. Then let

$$f(z) = u(x, y) + iv(x, y).$$

Then

$$\begin{cases} \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \\ \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0. \end{cases} \iff \lim_{z \to z_0} f(z) = w_0.$$

Proof. Assume that

$$\begin{cases} \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \\ \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0. \end{cases}$$

Let $\varepsilon > 0$. Then there is a δ_1 such that if $|(x, y) - (x_0, y_0)| < \delta_1 \Rightarrow |u(x, y) - u_0| < \varepsilon/2$ and a δ_2 such that if $|(x, y) - (x_0, y_0)| < \delta_2 \Rightarrow |v(x, y) - v_0| < \varepsilon/2$. Then let $\delta^* = \min(\delta_1, \delta_2)$. Consider $z : 0 < |z - z_0| < \delta^*$. Then

$$|f(z) - w_0| = \sqrt{(u(x, y) - u_0)^2 + (v(x, y) - v_0)^2} < \sqrt{\frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{4}} = \frac{\varepsilon}{\sqrt{2}} < \varepsilon.$$

Conversely, assume that $\lim_{z\to z_0} f(z) = w_0$. Exercise. □

Remark. What do we mean by $(x, y) \to (x_0, y_0)$? We simply mean the Euclidean 2-norm; This also preserves the idea of the norm or modulus in the complex plane.

Theorem 2.3.2. Let $z_0, w_1, w_2 \in \mathbb{C}$. Let f, g be functions such that $\lim_{z\to z_0} f(z) = w_1$ and $\lim_{z\to z_0} f(z) = w_2$. Then

(i) $\lim_{z\to z_0}(f+g)(z) = w_1 + w_2$

(ii) $\lim_{z\to z_0}(fg)(z) = w_1 w_2$

(iii) If $w_2 \neq 0$, then $\lim_{z\to z_0}\left(\frac{f}{g}\right)(z) = \frac{w_1}{w_2}$.

(iv) Let $\alpha, \beta \in \mathbb{C}$. Then $\lim_{z\to z_0}(\alpha f(z) + \beta) = \alpha w_1 + \beta$.

(v) $\lim_{z\to z_0} \overline{f(z)} = \overline{w}_1$

(vi) $\lim_{z\to z_0} |f(z)| = |w_1|$.

Proof. These results follow from the same results for two-variable real-valued functions. □

Definition 2.3.1 (Polynomial)**.** A *polynomial* is a function $p(z)$ of the form

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

for some $n \geq 0$ and $a_1, \ldots a_n \in \mathbb{C}$.

Corollary 2.3.3. If $p(z)$ is a polynomial function and $z_0 \in \mathbb{C}$, then

$$\lim_{z\to z_0} p(z) = p(z_0).$$

Proof. We can proceed by taking the limit of $f(z) = z$ and use induction to construct the polynomial. □

2.3.1 Limits at Infinity

In this section, we introduce a new point for the complex plane, denoted by ∞, called the point at infinity. We then can consider the *extended complex plane* to be

$$\mathbb{C} \cup \{\infty\}.$$

We can interpret this as the point we reach if we go very far from 0 in *any* direction. This is why there is no concept of $-\infty$. If we only consider one point as ∞, then we can consider the entirety of the complex plane as a sphere, with ∞ at the pole.

Definition 2.3.2. A (deleted) neighborhood of ∞ is the set

$$\{z \in \mathbb{C} : |z| > 1/\varepsilon\}$$

for some $\varepsilon > 0$. Informally put, you are very far away from 0. In this definition, ∞ is not included in the set, so there is no distinction between regular and deleted neighborhoods.

Definition 2.3.3. Let $z_0, w_0 \in \mathbb{C}\cup\{\infty\}$. Then let f be a function defined on a deleted neighborhood of z_0. We say that

$$\lim_{z\to z_0} f(z) = w_0$$

if

$$\forall B_\varepsilon(w_0)\ \exists B_\delta(z_0) \setminus \{z_0\} : f(B_\delta(z_0) \setminus \{z_0\}) \subseteq B_\varepsilon(w_0).$$

Remark. Let $z_0 \in \mathbb{C}$. Then $\lim_{z\to z_0} f(z) = \infty$ means that

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall z \in \mathbb{C},\ 0 < |z - z_0| < \delta \Rightarrow |f(z)| > \frac{1}{\varepsilon}.$$

The following theorem provides a way to change a limit that involves ∞ to a limit that does not involve infinity.

Theorem 2.3.4. Let $z_0, w_0 \in \mathbb{C}$. Then let f be a function. Then

(i) If $\lim_{z\to z_0} \frac{1}{f(z)} = 0 \iff \lim_{z\to z_0} f(z) = \infty$.

(ii) If $\lim_{z\to 0} f(1/z) = w_0$ then $\lim_{z\to\infty} f(z) = w_0$.

(iii) If $\lim_{z\to 0} \frac{1}{f(1/z)} = 0 \iff \lim_{z\to\infty} f(z) = \infty$.

Proof.

(i) Let $\varepsilon > 0$. There is a $\delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow \left|\frac{1}{f(z)} - 0\right| < \varepsilon$. This implies then that $|f(z)| < 1/\varepsilon$.

(ii) exercise

(iii) exercise

□

Example. Suppose we wanted to find

$$\lim_{z\to -1} \frac{iz+3}{z+1}.$$

Clearly, the issue is in the denominator, and so we would expect this limit to be infinite. Then by (i), we know that this limit is ∞ since the limit of the inverse is 0.

2.4 Continuity

Definition 2.4.1 (Continuous Function)**.** Let f be a function and let $z_0 \in \mathbb{C}$. We say that f is continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

Remark. This definition contains three conditions. One, that $f(z_0)$ exists. Two, that $\lim_{z\to z_0} f(z)$ exists. And three, that these two quantities are equal.

Theorem 2.4.1 (Composition of Functions)**.** Let f be continuous at z_0, and let g be continuous at $f(z_0)$. Then $g \circ f$ is continuous at z_0.

Proof. The proof is the same as proving the result for real-valued functions. □

Theorem 2.4.2. If a function f is continuous at a point z_0, and $f(z_0) \neq 0$, then f is nonzero on a neighborhood of z_0.

Theorem 2.4.3. Let $f(z) = u(x, y) + iv(x, y)$. Let $z_0 = x_0 + iy_0 \in \mathbb{C}$. Then

$$u, v \text{ are continuous at } (x_0, y_0) \iff f \text{ is continuous at } z_0$$

Proof. This follows from theorem [THEOREM]. □

Definition 2.4.2 (Bounded)**.** Let $R \subseteq \mathbb{C}$ be a region of the complex plane. Then we say that R is *bounded* if there is an $M > 0$ such that

$$\forall z \in R,\ |z| \leq M.$$

Definition 2.4.3 (Limit point)**.** Let $R \subseteq \mathbb{C}$. A point $z \in \mathbb{C}$ is called a limit point of R if for any $\varepsilon > 0$, then for the neighborhood $B_\varepsilon(z)$,

$$B_\varepsilon(z) \cap R \neq \varnothing \text{ and } B_\varepsilon(z) \cap R^c \neq \varnothing$$

Definition 2.4.4. A set $R \subseteq \mathbb{C}$ is closed if it contains all its limit points.

Theorem 2.4.4. Let $R \subseteq \mathbb{C}$ be bounded and closed. If f is continuous on R, then there exists a constant $M \geq 0$ such that

$$\forall z \in R,\ |f(z)| \leq M$$

and there is equality for at least one such z.

2.5 Derivatives

Definition 2.5.1 (Derivative at a Point)**.** Let $z_0 \in \mathbb{C}$, and let f be a function defined in a neigborhood of z_0. We say that f is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Then we write

$$f'(z_0) = \frac{df}{dz}(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Remark. If we write $h = z - z_0$, we can rewrite $f'(z_0)$ as

$$f'(z) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

we can also write h as Δz, and take this difference as an infinitesimal limit.

Example. Let $f(z) = z^2$. Let $z, h \in \mathbb{C}$. Then by the definition above,

$$\lim_{z \to z_0} \frac{f(z+h) - f(z)}{h} = \frac{z^2 + 2zh + h^2 - z^2}{h} = 2z + h = 2z.$$

Example. Let $f(z) = \overline{z}$. Let $z_0 = 0$. Then

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\overline{z}}{z}.$$

However, this limit does not exist. If we approach 0 from the real axis, we get that this limit is 1, and from the imaginary axis, we get -1. Therfore, f is not differentiable at 0.1

Even if f is a nice (i.e., C^∞) function of $u(x, y)$ and $v(x, y)$, it may not be differentiable as a complex function. In particular, we have that in general, the converse is is not true.

Warning. Let $z = x_0 + iy_0$, and let $f(z) = u(x, y) + iv(x, y)$. Then

$$u \text{ and } v \text{ differentiable at } (x_0, y_0) \not\Rightarrow f \text{ differentiable at } z_0.$$

Theorem 2.5.1. Let f be differentiable at $z_0 \in \mathbb{C}$. Then f is continuous at z_0.

Proof. We have that

$$f(z) = f(z) - f(z_0) + f(z_0) = \frac{f(z) - f(z_0)z - z_0}{(}z - z_0) + f(z_0).$$

Therefore,

$$\lim_{z \to z_0} f(z) = f'(z_0) \times 0 + f(z_0) = f(z_0).$$

thus f is continuous at z_0. □

As in the real case, the converse of this theorem is false. For instance, $f(z) = \overline{z}$ is continuous everywhere but differentiable nowhere.

2.5.1 Differentiability Rules

We have seen that there are functions that are nice in x and y, but are not necessarily differentiable.

Theorem 2.5.2. Let $z \in \mathbb{C}$ and f, g be functions which are differentiable at z. Let $c \in \mathbb{C}$ be a constant. Then

(i) cf is differentiable at z, and $(cf)'(z) = cf'(z)$.

(ii) $f + g$ is differentiable at z, and $(f + g)'(z) = f'(z) + g'(z)$.

(iii) fg is differentiable at z, and $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

(iv) If $g(z) \neq 0$, then f/g is differentiable at z, and

$$\left(\frac{f}{g}\right)(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

Proof.

1. blh
2. blag
3. By the definition,

$$\begin{aligned} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} &= \frac{f(z+h)g(z+h) - f(z)g(z+h) + f(z)g(z+h) - f(z)g(z)}{h} \\ &= \frac{f(z+h) - f(z)}{h} g(z+h) + \frac{g(z+h) - g(z)}{h} f(z) \\ &= f'(z)g(z) + g'(z)f(z) \end{aligned}$$

   where the last equality results from limit algebra and the continuity of f and g.

□

Theorem 2.5.3. Let $n \geq 0$ be an integer. Then

$$\frac{d}{dz}(z^n) = nz^{n-1}$$

is true everywhere.

Proof. This proof proceeds by induction on n. □

In order to evaluate these complex derivatives, it can help to treat z just like a real variable x; many of the familiar theorems from calculus apply.

Lastly, we can develop an analogue of the chain rule for complex numbers. The statement is the same as the real case.

Theorem 2.5.4. Let $z_0 \in \mathbb{C}$. Let f be a function that is differentiable at z_0, and let g be differentiable at $f(z_0)$. Then $g \circ f$ is differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(f(z_0))f(z_0).$$

Proof. The idea is that we want to use the result that tells us that the composition of a continuous function is continuous. Assume that $f(z) \neq f(z_0)$.From the definition of the derivative,

$$\begin{aligned}\frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0} &= \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \cdot \frac{f(z) - f(z_0)}{z - z_0} \\ &= \varphi(f(z)) \cdot \frac{f(z) - f(z_0))}{z - z_0}.\end{aligned}$$

where the function φ is

$$\varphi(w) = \begin{cases} \frac{g(w)-g(w_0)}{w-w_0} & w \neq w_0 = f(z_0) \\ g'(w_0) & w = w_0. \end{cases}$$

Note that this is still true if $f(z) = f(z_0)$, since then the numerator will just be 0. Therefore, we can take the limit and show that φ is continuous at w_0.

$$\lim_{w \to w_0} \varphi(w) = \varphi(w_0).$$

On the other hand, f is continuous at z_0 since it is differentiable. In conclusion, we can use the result saying that the composition of two functions is continuous to show that

$$\lim_{z \to z_0} = \varphi(f(z)) = \varphi(f(z_0)) = \phi(w_0) = g'(w_0).$$

and the limit of the other part in our product is

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).$$

Thus, after doing some limit algebra, we have proven that

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

□

2.6 Cauchy-Riemann Equations

Theorem 2.6.1. Let f be a function differentiable at some point $z_0 = x_0 + iy_0$. We write $f(z) = u(x, y) + iv(x, y)$. Then u and v have partial derivatives at x_0, y_0 and

$$\begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \end{cases}$$

Moreover,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

Proof. We know that f is differentiable. Therefore we know that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0)$$

Take to be real, with $h = \delta \in \mathbb{R}$. Then

$$\begin{aligned}&= \frac{1}{\delta}\left(u(x_0 + \delta, y_0)\right) + iv(x_0 + \delta, y_0) - u(x_0, y_0) - iv(x_0, y_0) \\ &= \frac{u(x_0 + \delta, y_0) - u(x_0, y_0)}{\delta} + i\frac{v(x_0 + \delta, y_0) - v(x_0, y_0)}{\delta}\end{aligned}$$

Therefore, we can write

$$\frac{u(x_0+\delta, y_0)-u(x_0,y_0)}{\delta} = \Re\left(\frac{f(z_0+h)-f(z_0)}{h}\right)$$

so

$$\lim_{\delta\to 0}\frac{u(x_0+\delta, y_0)-u(x_0,y_0)}{\delta} = \frac{\partial u}{\partial x} = \Re(f'(z_0)).$$

Where we can take the limit, since $\Re(\cdot)$ is a continuous function. On the other hand, we have that

$$\lim_{\delta\to 0}\frac{v(x_0+\delta, y_0)-v(x_0,y_0)}{\delta} = \frac{\partial v}{\partial x} = \Im(f'(z_0)).$$

In particular,

$$f'(z_0) = f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$$

Now we do the same but with $h = 0 + i\delta$. □

If u and v have partial derivatives and the equation is true, then f is differentiable. However, the converse is not true; differentiability in the complex sense is *stronger* than differentiability in two real functions u, v. The interpretation of this is that partial derivatives of u, v existing only tells us about the existence of limits when we approach from the real or the imaginary axis.

However, with some extra assumptions, we can prove a partial converse to the statement.

Theorem 2.6.2. Let $z_0 = x_0 + iy_0 \in \mathbb{C}$. Let $f(z) = u(x,y) + iv(x,y)$ be a function defined on a neighborhood $B_r(z_0)$. Assume that

(i) u_x, u_y, v_x, v_y are defined everywhere on $B_r(z_0)$.

(ii) u_x, u_y, v_x, v_y are continuous at (x_0, y_0).

(iii) The Cauchy-Riemann equations are satisfied ($u_x = v_y$ and $u_y = -v_x$ at (x_0, y_0)).

Then f is differentiable at z_0 and

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0).$$

Proof. Assumptions (i) and (ii) imply that u and v are differentiable at (x_0, y_0) (note that this is stronger than the existence of partial derivatives). This implies that

$$\begin{aligned}
u(x,y) - u(x_0,y_0) &= u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \\
v(x,y) - v(x_0,y_0) &= v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y
\end{aligned}$$

where $\varepsilon_i \to 0$ when $\Delta x, \Delta y \to (0,0)$. By assumption (iii), we get that $v_y(x_0,y_0)\Delta y = u_x(x_0,y_0)$. We write $h = \Delta x + i\Delta y$

$$\begin{aligned}
f(z_0+h) - f(z_0) &= u(x,y) + iv(x,y) - u(x_0,y_0) + iv(x_0,y_0) \\
&= u_x(x_0,y_0)\Delta x - v_x(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \\
&- v_x(x_0,y_0)\Delta x - u_x(x_0,y_0)\Delta y - i\varepsilon_3\Delta x + i\varepsilon_4\Delta y \\
&= (u_x(x_0,y_0)_i v_x(x_0,y_0))(\Delta x + i\Delta y) + (\varepsilon_1 + i\varepsilon_3)\Delta x + (\varepsilon_2 + i\varepsilon_4)\Delta y
\end{aligned}$$

Then dividing by h,

$$\frac{f(z_0+h)-f(z_0)}{h} = u_x(x_0,y_0) + iv(x_0,y_0) + (\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{h} + (\varepsilon_1 + i\varepsilon_4)\frac{\Delta y}{h}$$

Then

$$\left|(\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{h}\right| = |\varepsilon_1 + i\varepsilon_3|\frac{|\Delta x|}{|\Delta x + i\Delta y} \leq |\varepsilon_1| + |\varepsilon_2| \to 0.$$

so

$$\lim_{h\to 0}(\varepsilon_1 + i\varepsilon_3)\frac{\Delta x}{h} = 0$$

and same for

$$\lim_{h\to 0}(\varepsilon_2 + i\varepsilon_4)\frac{\Delta y}{h} = 0$$

Therefore,

$$\lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h} = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Hence $f'(z_0)$ exists and equals $u_x(x_0, y_0) + iv_x(x_0, y_0)$. $\square$

2.7 Some Sufficent Conditions

How do you prove that a complex function is differentiable? There are several ways:

1. Use the definition.
2. Using differentiation rules (such as polynomials, quotients of polynomials, etc.) Works best if we are working with explicit functions.
3. The previous theorem we proved, Theorem [NUMBER]. Useful for functions expressed in terms of x and y.

Notation. For a complex number z, we write e^z to mean

$$e^z := e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

Example. Let $f(z) = e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$. The real and imaginary functions u, v are

$$u(x,y) = e^x\cos(y)$$
$$v(x,y) = e^x\sin(y)$$

u and v have partial derivatives

$$u_x = e^x\cos(y)$$
$$u_y = -e^x\sin(y)$$

and

$$v_x = e^x\sin(y)$$
$$v_y = e^x\cos(y)$$

These partial derivatives are continuous everywhere. Moreover, the Cauchy-Riemann equations hold everywhere. By the theorem, f is differentiable everywhere, and we know that its derivative is

$$f'(z) = e^x\cos(y) + ie^x\sin(y)$$

which also happens to be f. Therefore,

$$\frac{d}{dz}(e^z) = e^z.$$

Example. Let $f(z) = |z|^2 = x^2 + y^2$. So $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. The partial derivatives for u are

$$u_x = 2x$$
$$u_y = 2y.$$

and $v_x = v_y = 0$. These are all continuous everywhere. However, this does not satisfy the Cauchy-Riemann equations for any points except for the point $(0, 0)$ and $f'(0) = 0$. However, f is not differentiable at any $z \neq 0$.

2.8 Polar Coordinates

Not covered

2.9 Holomorphic Functions

Before we discuss Holomorphic functions, we need to define some topological concepts.

Definition 2.9.1 (Open, Connected, Domain)**.** Let $S \subseteq \mathbb{C}$.

1. We say that S is an *open set* if for any $z \in S$, there exists an $\varepsilon > 0$ such that $B_r(z) \subseteq S$. Equivalently, S does not contain its limit points.
2. We say that S is *disconnected* if it is the union of two disjoint open sets. Otherwise, it is said to be *connected*. Equivalently, a set S is connected if for any pair of elements z_1, z_2 in S can be connected by a polygonal line in S consisting of finitely many segments.
3. A set which is nonempty, open, and connected is called a *domain*.

Definition 2.9.2 (Holomorphic)**.** A function f is *holomorphic* at a point if it is differentiable on a neighborhood of a point z_0. A function f is *holomorphic* (or *analytic*) on an open set S if it is differentiable for any point in S. An *entire function* is function which is analytic on $\mathbb{C}$.

Example. Consider $f(z) = 1/z$. This is holomorphic on $\mathbb{C}^*$. A polynomial is an entire function. The function $f(z) = |f|^2$ is differentiable at 0, but *not* holomorphic anywhere.

Theorem 2.9.1. Let f, g be functions.

1. If f and g are holomorphic on an open set S, then
$$f + g, f - g, fg$$
are analytic on S. If $g \neq 0$ on S, then f/g is also holomorphic on S.
2. If f is holomorphic on an open set S, and g is holomorphic on an open set T such that $f(S) \subseteq T$, then
$$g \circ f$$
is holomorphic on S.

Proof. Consequence of differentiation rules. □

The following allows us to say that antiderivatives are unique up to an additive constant.

Theorem 2.9.2. Let $D \subseteq \mathbb{C}$ be a domain, and let f be holomorphic on D. If $f'(z) = 0$ for any $z \in D$, then f is constant on D.

Proof. Let $f(z) = u(x, y) + iv(x, y)$, and $0 = f'(z) = u_x(x, y) + iv_x(x, y) = v_y(x, y) - iu_y(x, y)$. If a complex number is 0 then its real and imaginary parts are also 0. Therefore $u_x = u_y = v_x = v_y = 0$ on D. Consider $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \in D$. Then we desire that $f(z_1) = f(z_2)$. Assume that the line segment between z_1 and z_2 is completely included in D. We can parameterize the line segment as a function of t:

$$\begin{cases} x(t) = x_1 + (x_2 - x_1)t \\ y(t) = y_1 + (y_2 - y_1)t \quad 0 \leq t \leq 1 \end{cases}$$

Then

$$\frac{d}{dt}\left(u(x(t), y(t))\right) = \frac{dx}{dt}\frac{\partial u}{\partial x}(x(t), y(t)) + \frac{dy}{dt}\frac{\partial u}{\partial y}(x(t), y(t))$$

where the partial derivatives are 0, since $x(t), y(t) \in D$. Therefore we have only a function of one variable where $u(x(t), y(t))$ is constant with respect to t, so $u(x(0), y(0)) = u(x(1), y(1))$ and thus $u(x_1, y_1) = u(x_2, y_2)$. The same thing holds for v. Therefore, $f(z_1) = f(z_2)$.

Since D is a domain, it is connected. Then there is a polygonal line made of finitely many segments $s_1, \cdots s_n$ such that the line is contained entirely within D. Then let $f(z_1) = f(z_2)$, ☐

Definition 2.9.3. Let f be a function and $z \in \mathbb{C}$. We say that z_0 is a *singular point* of f if f is not holomorphic at z_0 but f is holomorphic at some point in any neigborhood of z_0.

Example. The function $z = 0$ is not a singular point of $f(z) = z^2$ since f is holomorphic everywhere. However, $z = 0$ is a singular point of $f(z) = 1/z$ since f is analytic everywhere except 0. Lastly, $z = 0$ is not a singular point of $f(z) = |z|^2$ since it is only differentiable at 0 but not holomorphic anywhere.

We know that being differentiable in the complex sense is much stronger than differentiability in either u or v. However, the property of being holomorphic is much much stronger. In fact, if a function is holomorphic on D then it is infinitely differentiable on D. If f_1 and f_2 are holomorphic on D, and f_1 and f_2 take the same values on a segment included in D, then $f_1 = f_2$. Hence, there are few holomorphic functions. There are even stronger statements such as these, which just go to show how powerful the condition of being holomorphic is.

3 Elementary Functions

3.1 The Exponential

We have seen this function before, but now we will give a proper definition.

Definition 3.1.1. Let $z = x + iy \in \mathbb{C}$. We define

$$e^z := e^x \cdot e^{iy}.$$

This is sometimes denoted as $\exp(z)$.

If z is real $(y = 0)$, then we get the usual exponential function on $\mathbb{R}$.

Theorem 3.1.1. The function e^z is an entire function, and for any $z \in \mathbb{C}$,

$$\frac{d}{dz}(e^z) = e^z.$$

or $\exp'(z) = \exp(z)$.

Proof. We have already proved this last chapter, using the Cauchy-Riemann equations. □

Definition 3.1.2. Let $z_1, z_2 \in \mathbb{C}$. Then

$$\begin{aligned} e^{z_1} \cdot e^{z_2} &= e^{z_1+z_2} \\ \frac{e^{z_1}}{e^{z_2}} &= e^{z_1-z_2}. \end{aligned}$$

Proof. This can be proved by taking $z_1 = x_1 + iy_1$ and $z_2 = x_2 - iy_2$. Then

$$e^{z_1}e^{z_2} = e^{x_1}e^{x_2}e^{iy_1}e^{iy_2} = e^{z_1+z_2}.$$

The second equation is proved similarly. □

These results are expected, whereas the next one is a point of departure from what we know about the real exponential.

Theorem 3.1.2. Let $z, z' \in \mathbb{C}$, where $z = x + iy$ and $z' = x' + iy'$. Then

$$e^z = e^{z'} \iff \begin{cases} x = x' \\ y = y' + 2k\pi \quad k \in \mathbb{Z}. \end{cases}$$

Here, the exponential is not injective.

Proof. It follows from the fact that

$$re^{i\theta} = r'e^{i\theta'} \iff \begin{cases} r = r' \\ \theta = \theta' + 2k\pi \quad k \in \mathbb{Z}. \end{cases}$$

Then set $r = e^x$, $r' = e^{x'}$, and $\theta = y$, $\theta' = y'$. □

Theorem 3.1.3. Let $z = x + iy \in \mathbb{C}$. Then

$$|e^z| = e^x$$

and

$$\arg(e^z) = \{y + 2k\pi : k \in \mathbb{Z}\}.$$

Proof. For the first, we have that

$$|e^z| = |e^x e^{iy}| = |e^x| \times |e^{iy}| = |e^x| = e^x$$

since $e^x > 0$. For the second part, we write e^z in exponential form, so

$$e^z = e^x e^{iy} = re^{iy}$$

with $r = e^x$. Therefore, y is an argument. Then $\arg(e^z)$ is created by adding integer multiples of 2π to any other arguemnt. □

Remark. Note that $e^z \neq 0$ for any $z \in \mathbb{C}$. It can be positive, negative, real, or complex, but not 0.

3.2 The Logarithm

For real numbers, we define the logarithm to be the inverse of the exponential:

$$y = \log(x),\ x > 0.$$

However, this definition is contingent on the uniqueness of x as a solution to the equation $y = e^x$. We demonstrated that in the complex case, there is not a unique solution to the equation $e^w = z$. If we write $z = re^{i\theta}$, and $w = u + iv$, then

$$\begin{aligned} e^w = z &\iff e^{u+iv} = e^{\ln(r)+i\theta} \\ &\iff \begin{cases} u = \ln(r) \\ v = \theta + 2k\pi, \quad k \in \mathbb{Z}. \end{cases} \end{aligned}$$

.

Definition 3.2.1. A *set-valued function* is a map that assigns to each z in the domain a set of values. It is useful to think of such a function F as

$$F : \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

Remark. The function arg can be seen as a set-valued function.

Definition 3.2.2. The logarithm is the set-valued function log defined for $z \neq 0$ by

$$\log(z) := \ln|z| + i\arg(z)$$

That is, if $z = re^{i\theta}$ for $r > 0$ and $\theta \in \mathbb{R}$,

$$\log(z) = \ln r + i(\theta + 2k\pi),\ k \in \mathbb{Z}.$$

The *principal value* of the logarithm is

$$\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z),\ z \neq 0$$

Proposition 3.2.1. For $z \neq 0$, then

$$e^{\log(z)} = z.$$

For $z \in \mathbb{C}$, then $\log(e^z) = z + i2\pi k,\ k \in \mathbb{Z}$.

Proof. Let $z \in \mathbb{C}^*$. Then $z = re^{i\theta}$ with $r > 0$, $\theta \in \mathbb{R}$. Then

$$\log(z) = \ln(r) + i\left(\theta + 2\pi k\right),\ k \in \mathbb{Z}$$

Then for any $k \in \mathbb{Z}$,

$$e^{\ln(r)+i(\theta+2k\pi)} = e^{\log r}e^{i\theta}e^{i2\pi k} = re^{i\theta} = z.$$

Now let $z = x + iy \in \mathbb{C}$. Then

$$\begin{aligned} \log(e^z) &= \ln|e^z| + i\arg(e^z) \\ &= \ln(e^x) + i(y + 2\pi k),\ k \in \mathbb{Z} \\ &= x + iy + i2\pi k,\ k \in \mathbb{Z} \\ &= z + 2i\pi k, k \in \mathbb{Z}. \end{aligned}$$

□

Proposition 3.2.2. The function Log defined on $\mathbb{C}^*$ is the inverse function of exp restricted to the set

$$\{z \in \mathbb{C} : -\pi < \Im(z) \leq \pi\}.$$

Moreover, Log on $\mathbb{C} \setminus (\mathbb{R}^-)$ is the inverse function of exp restricted to the set

$$\{z \in \mathbb{C} : -\pi < \Im(z) < \pi\}$$

where $\mathbb{R}^- = (-\infty, 0]$.

Proof. Let $z = x + iy$ with $-\pi < y \leq \pi$. Then $\operatorname{Arg}(e^z) = y$. □

3.3 Branches and Logarithmic Derivatives

Our goal in this section is to differentiate the Logarithmic function. As you recall, it is a set-valued function, and not one with single values. Therefore, we will first show that we can differentiate the principal value of the logarithm.

We must first notice that Log is not continuous at $z \in \mathbb{R}^-$, since Arg is also not continuous on this set.

Theorem 3.3.1. The function Log is analytic on $\mathbb{C} \setminus \mathbb{R}^-$, and

$$\frac{d}{dz}\left(\operatorname{Log}(z)\right) = \frac{1}{z}.$$

Proof. We know that Log is continuous on $\mathbb{C} \setminus \mathbb{R}^-$, since $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$. The modulus is continuous on $\mathbb{C}^*$, and Arg is continuous on $\mathbb{C} \setminus \mathbb{R}^-$. Now consider two points, $z, z_0 \in \mathbb{C} \setminus \mathbb{R}^-$. Then, letting $w = \operatorname{Log}(z)$ and $w_0 = \operatorname{Log}(z_0)$,

$$\frac{\operatorname{Log}(z) - \operatorname{Log}(z_0)}{z - z_0} = \frac{w - w_0}{e^w - e^{w_0}}$$

We know that the quantity

$$\lim_{w \to w_0} \frac{e^w - e^{w_0}}{w - w_0} = e^{w_0}$$

so then

$$\lim z \to z_0 \frac{e^w - e^{w_0}}{w - w_0} = e^{w_0}$$

by the composition of limits. Since the right hand side is nonzero,

$$\lim_{z \to z_0} \frac{\log(z) - \operatorname{Log}(z_0)}{z - z_0} = \frac{1}{e^{w_0}} \tag{4}$$

$$= \frac{1}{z_0}. \tag{5}$$

as desired. □

What we did above corresponds to the following idea about set-valued functions.

Definition 3.3.1. A *branch* of a set-valued function f is a single-valued function F such that F is analytic on some domain $\mathcal{D}$, and, for any $z \in \mathcal{D}$, $F(z)$ is a value of $f(z)$.

It should be clear from the above definition that the function Log on $\mathbb{C} \setminus \mathbb{R}^-$ is a branch of the log function. Moreover, for any $\alpha \in \mathbb{R}$, we can define a branch F of log on $\{z = re^{i\theta} : r > 0, \alpha < \theta < \alpha + 2\pi\}$ by $F(re^{i\theta}) = \ln|r| + i\theta$ for $r > 0,\ \alpha < \theta < \alpha + 2\pi$, where we exclude the endpoints in order to make this branch analytic.

3.4 Identities Involving Logarithms

Definition 3.4.1. If $A, B \subseteq \mathbb{C}$, then

$$A + B = \{a + b : a \in A, andb \in b\}.$$

Similarly, for $\lambda \in \mathbb{C}$,

$$\lambda A = \{\lambda \cdot a : a \in A\}.$$

With this in mind, we have a meaninfgul way to linearly combine set-valued functions, since we can always return another set under these definitions. Therefore,

Proposition 3.4.1. Let $z_1, z_2 \in \mathbb{C}^*$. Then

$$\log(z_1 z_2) = \log(z_1) + \log(z_2)$$

and

$$\log\left(\frac{z_2}{z_2}\right) = \log(z_1) - \log(z_2)$$

Proof. Recall that

$$\begin{aligned}\log(z_1 z_2) &= \ln|z_1 z_2| + i\arg(z_1 z_2)\\ &= \ln|z_1|\ln|z_2| + i(\arg(z_1) + \arg(z_2))\\ &= \log(z_1) + \log(z_2).\end{aligned}$$

We can do a similar proof for $\log(z_1/z_2)$, whereby the argument of a quotion is a subtraction. $\square$

Warning. Note that $\mathrm{Log}(z_1 z_2)$ can be different from $\mathrm{Log}(z_1)+\mathrm{Log}(z_2)$. Moreover, $\log(z^2) \neq 2\log(z)$ since $2\log(z) \neq \log(z) + \log(z)$.

3.5 Power Functions

Recall that for $x > 0$, $a \in \mathbb{R}$, we have that $x^a = e^{a\ln(x)}$. Then we can extend this definition to the complex plane.

Definition 3.5.1. Let $c \in \mathbb{C}$. The power function with exponent c is defined for any complex number $z \neq 0$ by

$$z^c = e^{c\log(z)}.$$

In general, this is a multiple valued function due to the $\log(z)$ in the exponent. Now, lets see how this definition compares to our previous definitions. If $z = re^{i\theta}$, where $r > 0$ and $\theta \in \mathbb{R}$. then we can write

$$z^c = e^{c\ln(r)}e^{ic(\theta+2k\pi)},\ k \in \mathbb{Z}.$$

Note that $e^{i2kc\pi}$ is not necessarily an integer. If $c = n \in \mathbb{Z}$, then

$$z^n = e^{n\ln(r)}e^{in\theta}e^{i2k\pi n} = r^n e^{in\theta}$$

which is a familar result, so this definition does not appear to contradict our previous notions. Thus, restricted to the integers, z^n is a single-valued function.

If $c = 1/n$, then recall that we attain n roots; thus

$$\begin{aligned}z^{1/n} &= e^{\ln(r)/n}e^{i(\theta+2\pi k)/n},\ k \in \mathbb{Z}\\ &= (r)^{1/n}e^{i\left(\frac{\theta+2\pi k}{n}\right)},\ 0 \leq k < n\end{aligned}$$

so we get n different values which are the nth roots of z. Thus, we have finitely many values for $c \in \mathbb{Q}$.

Finally, if $c \in \mathbb{C} \setminus \mathbb{Q}$, then there are infinitely many values for the expression.

Warning. From now on, x^c means the multiple valued powere even for $x > 0$. For instance, $r^{1/n} \neq \sqrt[n]{r}$. The former are the n nth roots of $x \in \mathbb{C}$, whereas $\sqrt[n]{x}$ is the single positive nth root of x. More generally, if $a \in \mathbb{R}$, then

$$x^a \neq e^{a \ln(x)}$$

since the left has several values if $a \neq \mathbb{Z}$, whereas the right has only one positive real number.

Definition 3.5.2. Let $c \in \mathbb{Z}$. For $z \neq 0$, the principal value of z^c is denoted by

$$\operatorname{PV}(z^c) = e^{c \operatorname{Log}(z)}.$$

Corollary 3.5.1. For a positive integer n, then $\operatorname{PV}(z^{1/n})$ is the principal nth root of z.

Moreover, we can modify our previous definition of exponentiation,

$$z^c = e^{c \log(z)} = e^{c(\operatorname{Log}(z)+2i\pi k)} = (\operatorname{PV}(z^c))\, e^{i2\pi kc}$$

Proposition 3.5.2. Let $z \in \mathbb{C}$, and $z \in \mathbb{C}^*$. Then

$$\frac{1}{z^c} = z^{-c}, \ \frac{1}{\operatorname{PV}(z^c)} = \operatorname{PV}(z^{-c}).$$

Proof. We have that

$$\frac{1}{z^c} = \frac{1}{e^{c(\operatorname{Log}(z)+i2\pi k)}} = e^{-c(\operatorname{Log}(z)+i2\pi k)} = z^{-c}$$

And it easily follows from definition that

$$\frac{1}{\operatorname{PV}(z^c)} = \frac{1}{e^{c \operatorname{Log}(z)}} = e^{-c \operatorname{Log}(z)} = \operatorname{PV}(z^{-c}).$$

□

Proposition 3.5.3. Let $c \in \mathbb{C}$, and $z \in \mathbb{C}^*$, and $n \in \mathbb{Z}$. Then

$$z^c z^n = z^{c+n}.$$

Moreover,

$$\operatorname{PV}(z^c)\operatorname{PV}(z^n) = \operatorname{PV}(z^{c+n}).$$

Proof. We know that z^n has only one value. Thus

$$\begin{aligned} z^c z^n = e^{c(\operatorname{Log}(z)+i2\pi k)} e^{n \operatorname{Log}(z)} \\ &= e^{c(\operatorname{Log}(z)+i2\pi k)} e^{n(\operatorname{Log}(z)+i2\pi k)} \\ &= e^{(c+n)(\operatorname{Log}(z)+i2\pi k)} &= z^{c+n}. \end{aligned}$$

The proof of the second is true, and the proof is easy since these are single-valued functions. □

Remark. In general, for $c, d \in \mathbb{C}$, $z^c z^d \neq z^{c+d}$.

3.5.1 Differentiating Power Functions

Theorem 3.5.4. Let $c \in \mathbb{C}$. Then $\mathrm{PV}(z^c)$ is a holomorphic function on $\mathbb{C} \setminus \mathbb{R}^-$. Its derivative is

$$\frac{d}{dz}\,\mathrm{PV}(z^c) = c\,\mathrm{PV}(z^{c-1}).$$

Proof. We jave that $\mathrm{PV}(z) = e^{c\,\mathrm{Log}(z)}$. We know that $c\,\mathrm{Log}(z)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}^-$, and exp is holomorphic on $\mathbb{R}$. By the chain rule, $\mathrm{PV}(z^c)$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}^-$, and

$$\begin{aligned}
\frac{d}{dz}\,\mathrm{PV}(z^c) &= \frac{d}{dz}(c\,\mathrm{Log}(z))e^{c\,\mathrm{Log}(z)} \\
&= \frac{c}{z}\,\mathrm{PV}(z^c) \\
&= c\,\mathrm{PV}(z^{-1})\,\mathrm{PV}(z^c) \\
&= c\,\mathrm{PV}(z^{c-1}).
\end{aligned}$$

□

3.6 Trigonometric Function

Recall that if $x \in \mathbb{R}$, we get that

$$\begin{aligned}
e^{ix} &= \cos x + i \sin x \\
e^{-ix} &= \cos x - i \sin x
\end{aligned}$$

If we combine the two equations, we see that

$$\begin{aligned}
\cos x &= \frac{e^{ix} + e^{-ix}}{2} \\
\sin x &= \frac{e^{ix} - e^{-ix}}{2i}
\end{aligned}$$

We will use these functions to extend these functions to the comlex plane.

Definition 3.6.1. For $z \in \mathbb{C}$, we define

$$\begin{aligned}
\cos z &= \frac{e^{iz} + e^{-iz}}{2} \\
\sin z &= \frac{e^{iz} - e^{-iz}}{2i}.
\end{aligned}$$

Definition 3.6.2. The functions sin and cos are entire functions, and for $z \in \mathbb{C}$,

$$\begin{aligned}
\frac{d}{dz}\cos(z) &= -\sin z \\
\frac{d}{dz}\sin(z) &= \cos z
\end{aligned}$$

Proof. This can be proved via the chain rule, since we know how to differentiate the exponential. □

Many trigonometric formulas are stil valid, even after we have extended these functions into the complex plane. In particualr,

$$\cos^2 z + \sin^2 z = 1.$$

One way of proving this is that the left hand function is analytic on the complex plane, and so is 1. If two analytic functions are the same on a line segment, then they are the same everywhere, and so we can conclude they are the same.

Recall that $\sinh(y)$ and $\cosh(y)$ have exponential formulas.

Definition 3.6.3. For $z \in \mathbb{C}$, the hyperbolic functions are

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}$$

Then we have that

$$\begin{cases} \sin(iz) = i\sinh(z) \\ \cos(iz) = \cosh(z). \end{cases}$$

3.7 Zeroes and Singularities of Trigonometric Functions

Definition 3.7.1 (Zero of a function)**.** A *zero of a function* f is a point z such that

$$f(z) = 0.$$

The quintessential example is that $f(x) = x^2 + 1$ has no zeroes in $\mathbb{R}$, but $f(z) = z^2 + 1$ has two on $\mathbb{C}$.

Theorem 3.7.1. The zeroes of the function sin on $\mathbb{C}$ are $\{k\pi : k \in \mathbb{Z}\}$. The zeroes of cos on $\mathbb{C}$ are $\{k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\}$. This means that they can only be zero on the real line.

Proof. Let $z \in \mathbb{C}$, where $z = x + iy$. Then

$$\begin{aligned} \sin z = 0 &\iff \frac{e^{iz} - e^{-iz}}{=} 0 \\ &\iff e^{iz} = e^{-iz} \\ &\iff e^{-y+ix} = e^{y-ix} \\ &\iff \begin{cases} -y = y \\ x = -x + 2\pi k_0, \ k_0 \in \mathbb{Z}. \end{cases} \qquad \iff \begin{cases} y = 0 \\ x = k_0\pi, \ k_0 \in \mathbb{Z}. \end{cases} \end{aligned}$$

We can provide a similar proof for cos. □

Definition 3.7.2. For $z \in \mathbb{C} \setminus \{k\pi + \frac{\pi}{2} : k \in \mathbb{Z}\}$, then

$$\tan(z) = \frac{\sin(z)}{\cos(z)}.$$

For $z \in \mathbb{C} \setminus \{k\pi : k \in \mathbb{Z}\}$, we define $\cot(z)$ as

$$\cot(z) = \frac{\cos(z)}{\sin(z)}.$$

4 Contour Integration

Our goal is to define and study integrals of functions of a complex variable along curves. There are some very nice and powerful properties; indeed, the theory we develop here can greatly simplify integration of real functions by seeing them as complex.

4.1 Derivatives of Functions $w(t)$

We consider complex valued functions of a real variable.

Definition 4.1.1. A complex valued function of a real variable $w : I \to \mathbb{C}$ where I is a real interval.

We write $w(t) = u(t) + iv(t)$, with $u, v : I \to \mathbb{R}$.

Definition 4.1.2. Let $w : I \to \mathbb{C}$ be defined on a real interval I, written as $w(t) = u(t) + iv(t)$, and let $t_0 \in I$. We say that w is *continuous* at t_0 if u, v are continuous at t_0.

Moreover, w is differentiable at t_0 if u and v are differentiable at t_0. Then the derivative is $w'(t_0) = u'(t_0) + iv'(t_0)$.

Proposition 4.1.1. Let $w_1, w_2 : I \to \mathbb{C}$, and assume that they are both differentiable at some point t_0 in the interval. Then the following are true:

1. $w_1 + w_2$ is differentiable at t_0 and $(w_1 + w_2)'(t_0) = w_1'(t_0) + w_2'(t_0)$.
2. $w_1 w_2$ is differentiable at t_0 and $(w_1 w_2)'(t_0) = w_1'(t_0)w_2(t_0) + w_(t_0)w_2'(t_0)$.
3. If $w_2(t) \neq 0$, then
$$\left(\frac{w_1}{w_2}\right)'(t_0) = \frac{w_1'(t_0)w_2(t_0) - w_1(t_0)w_2'(t_0)}{w_2(t_0)^2}.$$

The above should be verified without difficulty by applying known differentiation rules. However, we have two possible chain rules in our differentiation; we can compose w with a function $f : \mathbb{R} \to \mathbb{R}$, or we can compose w with another function $\mathbb{C} \to \mathbb{C}$.

Theorem 4.1.2 (Chain Rule 1)**.** Let $w : I \to \mathbb{C}$ and $g : J \to \mathbb{R} >$ where I, J are real intervals. Let $t_0 \in J$, and assume g is differentiable at t_0 and w is differentiable at $g(t_0)$. Then $(w \circ g)(t_0) = w(g(t))$ and the derivative is given by $w'(g(t))g'(t)$.

Proof. If we write $w(t) = u(t) + iv(t)$, then the proof follows immediately. □

Theorem 4.1.3. Let $w : I \to \mathbb{C}$, where I is a real interval. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex-valued function of a complex variable. Let $t \in I$. Assume w is differentiable at t_0, and f is (complexly) differentiable at $w(t_0)$. Then $f \circ w$ is differentiable at t_0, and

$$(f \circ w)'(t_0) = f'(w'(t_0))w'(t_0).$$

Proof. Write $w(t) = u(t) + iv(t)$, and write $f(z) = U(x, y) + iV(x, y)$. Then

$$f(w(t_0)) = U(u(t_0), v(t_0)) + iV(u(t_0), v(t_0))$$

Differentiating via the chain rule for two-variable functions we get

$$\begin{aligned}
\frac{d}{dt} f(w(t_0)) &= u'(t)\frac{\partial U}{\partial x} + v'(t)\frac{\partial U}{\partial y} + i\left(u'(t)\frac{\partial V}{\partial x} + v'(t)\frac{\partial V}{\partial y}\right) \\
&= u'(t_0)\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial x}\right) + v'(t_0)\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial y}\right) \\
&= u'(t)f'(t_0) + v'(t)(if'(t_0))
\end{aligned}$$

where we get the last step by the Cauchy-Riemann equations and substituting $U_y = -V_x$ and $U_x = V_y$. Therefore, we get

$$\frac{d}{dt} f(w(t)) = f'(w(t_0))w'(t_0)$$

□

The reason this works is due to the Cauchy-Riemann equations and the holomorphicity of functions.

4.2 Definite Integrals of Functions $w(t)$

Definition 4.2.1. Let $w : I \to \mathbb{C}$, with $w(t) = u(t) + iv(t)$, and assume that I has endpoints a and b which are possibly $\pm\infty$. If the integrals

$$\int_a^b u(t)dt$$
$$\int_a^b v(t)dt$$

Then we define

$$\int_a^b w(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

It follows from the definition that

$$\Re\left(\int_a^b w(t)dt\right) = \int_a^b \Re(w(t))dt$$
$$\Im\left(\int_a^b w(t)\right) = \int_a^b \Im(w(t))dt.$$

Proposition 4.2.1. Let $a < b < c$ be real values. Let w be a complex-valued function of a real variable. If

$$\int_a^b w(t)dt$$
$$\int_b^c w(t)dt$$

exist, then

$$\int_a^c w(t)dt$$

exists and equals

$$\int_a^b w(t)dt + \int_b^c w(t)dt$$

Theorem 4.2.2 (Fundamental Theorem of Calculus)**.** Let $a < b$ be real numbers. Let $w : [a, b] \to \mathbb{C}$ be a continuous function, and assume that there is a function $W : [a, b] \to \mathbb{C}$ such that W is differentiable on $[a, b]$ and $W'(t) = w(t)$. Then

$$\int_a^b w(t)dt = [W(t)]_a^b = W(b) - W(a).$$

4.3 Contours

Definition 4.3.1 (Path)**.** A path γ is a continuous function $[a, b] \to \mathbb{C}$ for some $a < b$. We write C as $z(t)$, where $t \in [a, b]$.

We call γ a simple arc if it does not intersect itself, or $z(t_1) \neq z(t_2)$ if $t_1 \neq t_2$.

We call γ a simple closed curve if it is simple, except $z(a) = z(b)$.

A simple closed curve γ is *positively-oriented* if it is drawn in the counter clockwise direction.

Definition 4.3.2 (Reparameterization)**.** Let γ be a path given by $\gamma = \gamma(t)$, $a \leq t \leq b$. A reparameterization of γ is a parameterization of the form $\Gamma(\tau) = \gamma(\Phi(\tau))$, where $\alpha \leq \tau \leq \beta$ for some function $\Phi : [\alpha, \beta] \to [a, b]$ which is C^1, bijective, and $\Phi'(\tau) > 0$.

We now say that if an arc can be obtained from another one by a reparameterization, then we consider these arcs to be the same (or belonging to the same equivalence class). Thus, an arc is defined by a parameterization $\gamma = \gamma(t)$, up to a parameterization. We have that $\gamma(t) = \Gamma(\tau)$. This has the effect of allowing us to arbitrarily choose the interval on which the parameterization is defiend.

Definition 4.3.3 (Differentiable Path)**.** A path γ given by $\gamma = \gamma(t)$, $a \leq t \leq b$ is differentiable if $\gamma(t)$ is C^1 on $[a, b]$.

Definition 4.3.4 (Path Length)**.** If a path γ is differentiable, we define the length of $\gamma = \gamma(t)$, $a \leq t \leq b$ to be

$$\int_a^b |\gamma'(t)| \, dt.$$

For such a definition, we can check that length does not depend on the parameterization; for $\Gamma(\tau) = \gamma(\Phi(\tau))$, we see that

$$\begin{aligned} \int_\alpha^\beta |\Gamma'(\tau)| \, d\tau &= \int_\alpha^\beta |\gamma'(\Phi(\tau))\Phi'(\tau)| \, d\tau \\ &= \int_a^b |\gamma'(t)| \, dt \end{aligned}$$

after executing the change of variables $t = \Phi(\tau)$.

Definition 4.3.5 (Smooth Path)**.** A path given by $\gamma(t)$ is smooth if it is differentiable and $\gamma'(t) \neq 0$.

With all this, we can define what a contour is.

Definition 4.3.6 (Contour)**.** Let γ be a path given by $\gamma = \gamma(t)$, $a \leq t \leq b$. We say that γ is a contour if it is a simple closed path which is piecewise C^1.

It can be tricky to keep track of the meanings of different

4.4 Integration and Operations on Contours

Definition 4.4.1 (Contour Integral)**.** Let $\gamma : [a, b] \to U \subseteq \mathbb{C}$ be a contour and let $f : \mathbb{C} \to \mathbb{C}$ be piecewise continuous. Then we define the *contour integral of f along γ* as

$$\int_\gamma f(z) dz := \int_a^b f(\gamma(t))\gamma'(t) dt$$

We can check that this definition does not depend on reparameterization, by using a similar argument as in last section. Next, we have the following properties:

Proposition 4.4.1. Let f, g be piecewise C^1 on a contour γ. Let $c_1, c_2 \in \mathbb{C}$. Then

$$\int_\gamma (c_1 f + c_2 g)(z) dz = c_1 \int_\gamma f(z) dz + c_2 \int_\gamma g(z) dz.$$

Definition 4.4.2. Let γ be a path. Then we define

$$-\gamma := \gamma(-t).$$

Theorem 4.4.2. Let $\gamma, \gamma_1, \gamma_2$ be contours. If f is piecewise continuous on γ, then

$$\int_{-\gamma} f = -\int_\gamma f.$$

Moreover, if f is piecewise on γ_1 and γ_2, and $\gamma_1 + \gamma_2$ is defined, then

$$\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

Suppose we have a contour, which is a a simple, closed path which is piecewise C^1. For instance, we can define the contour

$$\gamma(t) = \begin{cases} e^{it} & 0 \leq t \leq \pi \end{cases}$$

Note that $\gamma'(t)$ is always tangent to the curve at $\gamma(t)$.

4.4.1 Example with Branch Cuts

Example. Let $f(z) = \mathrm{PV}(z^{1/2})$ for $z \neq 0$. Such a function is holomorphic on $\mathbb{C} \setminus \mathbb{R}^-$. Suppose our contour is given by $\gamma(\theta) = e^{i\theta}$, and $0 \leq \theta \leq 2\pi$. Then we can try and find the integral along the contour:

$$\begin{aligned} f(\gamma(\theta)) &= \exp\left(\frac{1}{2}\operatorname{Log}(\gamma(\theta))\right) \\ &= \exp\left(\frac{1}{2}\left(\ln|\gamma(\theta)| + i\operatorname{Arg}(\gamma(\theta))\right)\right) \\ &= \exp\left(\frac{i}{2}\operatorname{Arg}(\gamma(\theta))\right). \end{aligned}$$

However, we must note that $\operatorname{Arg}(\gamma(\theta))$ is θ if $\theta \in [0, \pi]$, and $\theta - 2\pi$ if $\theta \in (\pi, 2\pi]$. Then

$$f(\gamma(\theta)) = \begin{cases} e^{i\theta/2} & 0 \leq \theta \leq \pi \\ -e^{i\theta/2} & \pi < \theta \leq 2\pi. \end{cases}$$

This is piecewise continuous, so

$$\begin{aligned} \int_\gamma f(z) dz &= \int_0^\pi e^{i\theta/2} i e^{i\theta} d\theta + \int_\pi^{2\pi} -e^{i\theta/2} i e^{i\theta/2} d\theta \\ &= \left[\frac{2}{3} e^{3i\theta/2}\right]_0^\pi - \left[\frac{2}{3} e^{3i\theta/2}\right]_\pi^{2\pi} \\ &= -\frac{4}{3} i \end{aligned}$$

4.5 Bounding Integrals

In this section, we will develop some elementary ideas on how to properly bound integrals.

Lemma 4.5.1. Let $w : [a, b] \to \mathbb{C}$. Then

$$\left| \int_a^b w(t)dt \right| \leq \int_a^b |w(t)|dt.$$

Proof. We can write the intgral in exponential form, $r_0 e^{i\theta_0}$. Then

$$\begin{aligned} \left| \int_a^b w(t)dt \right| &= r_0 \\ &= e^{-i\theta_0} \int_a^b w(t)dt \\ &= \int_a^b e^{-i\theta_0} w(t)dt \end{aligned}$$

However, since r_0 is a real number,

$$\begin{aligned} r_0 = \Re(r_0) = \Re\left(\int_a^b e^{-i\theta_0} w(t)dt \right) \\ &= \int_a^b \Re\left(e^{-i\theta_0} w(t)dt \right) \\ &\leq \int_a^b |w(t)|dt \end{aligned}$$

□

Theorem 4.5.2. Let γ denote a contour of length L, and let f be a piecewise continuous function on γ. Assume that there is a constant $M \geq 0$ such that $|f(z)| \leq M$ for any z on γ. Then

$$\left| \int_\gamma f(z)dz \right| \leq ML$$

Proof. We assume that γ is givven by $\gamma = \gamma(t)$ for $t \in [a, b]$. Then

$$\begin{aligned} \left| \int_\gamma f(z)dz \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t)dt \right| \\ &\leq \int_a^b |f(\gamma(t))\gamma'(t)dt| \\ &\leq M \int_a^b |\gamma'(t)|dt \\ &= ML. \end{aligned}$$

□

4.6 Antiderivatives

Definition 4.6.1 (Antiderivatives)**.** Let f be a continuous function on a domain $\mathcal{D}$. THen we say that F is an *antiderivative* of f on $\mathcal{D}$ if F is holomorphic on $\mathcal{D}$ and $F'(z) = f(z)$ for any $z \in \mathcal{D}$.

We saw examples where the integral between two points does not depend on the contour we chose between them. Then we can prove the following result.

Theorem 4.6.1. Suppose that f is continuous on a domain $\mathcal{D}$. Then the following are equivalent.

(i) f has an antiderivative F.

(ii) The integral of f along a contour $\gamma \subset \mathcal{D}$ depends only on the initial and final point of γ.

(iii) The integral of f along a closed contour $\gamma \subset \mathcal{D}$ is 0.

That is, if these statements are true, then for any $z_1, z_2 \in \mathcal{D}$, for any contour γ from z_1 to z_2 included in $\mathcal{D}$, then

$$\int_\gamma f(z)dz = [F(z)]_{z_1}^{z_2} = F(z_1) - F(z_2).$$

Remark. An antiderivative, if it exists, is unique up to an additive constant. In other words, if F is an antiderivative of f on $\mathcal{D}$, then the antiderivatives of f on $\mathcal{D}$, then the antiderivatives are $F(z) + c$ for any constant $c \in \mathbb{C}$.

Proof. Clearly, $F(z) + c$ is an antiderivative of f. Then if G is an antiderivative of f on $\mathcal{D}$, then

$$(F - G)'(z) = F'(z) - G'(z) = f(z) - f(z) = 0.$$

So $F - G$ is holomorphic on the domain $\mathcal{D}$ with derivative 0, so $F - G$ must be constant. It is important that $\mathcal{D}$ is connected. □

How do we use the above theorem? We mostly use it to compute $\int_\gamma f$ when one already knows an antiderivative of f on a domain including γ.

Example. Let us find the integral

$$\int_\gamma \cos(z)dz$$

On a contour from -1 to 1. The shape of the contour does not matter, so we have that

$$F(z) = \sin z$$

which is holomorphic on the entire complex plane, or where $\mathcal{D} = \mathbb{C}$. Hence,

$$\sin(1) - \sin(-1) = 2\sin(1).$$

Example. Let us try to compute

$$\int_\gamma \frac{1}{z^2}dz$$

where γ is any contour from i to 1 but not containing 0. Then $\mathcal{D} = \mathbb{C}^*$, and the antiderivative of $1/z^2$ is $-1/z$. Then we get

$$\left[-\frac{1}{z}\right]_i^1 = -1 - i.$$

Example. One example where the above may fail is for $f(z) = 1/z$. This is continuous on $\mathcal{D} = \mathbb{C}^*$. Then we can consider γ_1 to be the positively oriented semicircle of radius 1 on the upper half of the real line, and let γ_2 be the negatively-oriented semicircle on the lower half of the plane. Then we have

$$\int_{\gamma_1} \frac{1}{z} dz = i\pi \neq -i\pi = \int_{\gamma_2} \frac{1}{z} dz.$$

The reason the theorem fails here since $1/z$ has no antiderivative defined on $\mathbb{C}^*$. What is true is that $1/z$ has an antiderivative on $\mathbb{C} \setminus \mathbb{R}^-$, which is $\mathrm{Log}(z)$.

Recall that the definition of the antiderivative necessarily means that the antiderivative is holomorphic on the given domain $\mathcal{D}$.

Now that we have played around with the result, we will provide a rigorous proof.

Proof.

(1) (i)$\Rightarrow$(ii). Suppose there is an antiderivative of f on $\mathcal{D}$. Let γ be a contour from z_1 to z_2 included in $\mathcal{D}$. In particular, we assume γ is a smooth path. Then

$$\begin{aligned}\int_\gamma f(z) dz &= \int_a^b f(\gamma(t))\gamma'(t) dt \\ &= F(z(b)) - F(z(a)) \\ &= F(z_2) - F(z_1).\end{aligned}$$

where we can invoke the Fundamental theorem of Calculus since f and γ' are continuous. For the general case, we can apply the particular case to each smooth piece. Therefore, we will get the desired result by summing the resultant integrals.

(2) (ii)$\Rightarrow$(i). Assume that the integral depends only on the endpoints. Then fix an arbitrary $z_0 \in \mathcal{D}$. Define

$$F(z_1) = \int_{z_0}^{z_1} f(z) dz$$

for any $z_1 \in \mathcal{D}$. This means the intgral along any contour from z_0 to z_1 included in $\mathcal{D}$, which is from assuming 2. Now we want to prove that

$$\lim_{\eta \to 0} \frac{F(z_1 + \eta) - F(z_1)}{\eta}.$$

Let $\varepsilon > 0$, let $\eta \in \mathbb{C} : 0 < |\eta| < \delta$. Because $\mathcal{D}$ is open by definition, there is a neighborhood of z_1 of radius δ_1 included in $\mathcal{D}$. Then for $|\eta| < \delta_1$, $z_1 + \eta$ is in $\mathcal{D}$ and

$$\begin{aligned}F(z_1 + \eta) - F(z_1) &= \int_{z_0}^{z_1+\eta} f(z) dz - \int_{z_0}^{z_1} f(z) dz \\ &= \int_{z_1}^{z_1+\eta} f(z) dz \\ &= \int_\gamma f(z) dz\end{aligned}$$

with $\gamma : \gamma(t) = z_0 + t\eta$, where $t \in [0, 1]$. Then we can write the above as

$$F(z_1 + \eta) - F(z_1) = \int_0^1 f(z_1 + t\eta)\eta \; dt.$$

Now let us compute the ratio

$$\begin{aligned}\frac{F(z_1+\eta)-F(z_1)}{\eta}-f(z_1) &= \int_0^1 f(z_1+t\eta)dt - f(z_1)\\ &= \int_0^1 \left(f(z_1+t\eta)-f(z_1)\right)dt.\end{aligned}$$

Then since f is continuous, then $\exists\delta_2$ such that $0<|z-z_1|<\delta_2 \Rightarrow |f(z)-f(z_1)|<\varepsilon$. Then we select $\delta=\min(\delta_1,\delta_2)$. Then we choose η such that $0<|\eta|<\delta$. Then the above implies that

$$\begin{aligned}\left|\frac{F(z_1+\eta)-F(z_1)}{\eta}-f(z_1)\right| &= \left|\int_0^1 (f(z_1+\eta)-f(z_1))dt\right|\\ &\le \int_0^1 |f(z_1+t\eta)-f(z_1)|\,dt\\ &< \varepsilon.\end{aligned}$$

This proves the desired result.

(3) (ii)⇒(iii). Assume (ii) is true. Let γ be a closed contour included in $\mathcal{D}$ with endpoint z_0. We weant to prove that $\int_\gamma f(z)dz=0$. Let $\gamma_0:\gamma(t)=z_0$. Then

$$\int_{\gamma_0} f(z)dz = \int f(z_0)\gamma'(t)dt = 0.$$

(4) (iii)⇒(ii). Let γ_1,γ_2 be contours from z_1 to z_2 included in the domain $\mathcal{D}$. Then we want to prove the two integrals are equal. Define $\gamma=\gamma_1-\gamma_2$. By (iii), we know the integral is 0. But we know that

$$\begin{aligned}\int_\gamma f &= \int_{\gamma_1} f + \int_{-\gamma_2} f\\ &= \int_{\gamma_1} f - \int_{\gamma_2} f\\ &= 0\end{aligned}$$

So

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

as desired.

□

4.7 The Cauchy-Goursat Theorem

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Theorem 4.7.1 (Jordan Curve Theorem)**.** sec. 43: If γ is a simple, closed curve, then there are two domains I and E such that I is in the inside of the contour and E is outside of it. Morever, I is bounded and bounded and called the interior of γ; E is unbounded and called the exterior. The points on γ are the boundary points of I and E.

$$I\cup\gamma\cup E=\mathbb{C}.$$

Theorem 4.7.2 (Cauchy-Goursat)**.** Let γ be a simple, closed contour. Let f a function holomorphic on a domain $\mathcal{D}$. If γ and its interior are in $\mathcal{D}$, then

$$\int_\gamma f(z)dz = 0.$$

Example. Consider $f(z) = 1/z$, which is holomorphic on $\mathbb{C}^*$. Let γ be a simple closed contour such that the countour and its interior are contianed in $\mathbb{C}^*$. Then the integral

$$\int_\gamma \frac{1}{z}dz = 0.$$

The first proof was in the case where f' is continuous on $\mathcal{D}$. In this case, it relies on Green's theorem, which tells us how to relate an integral on a domain to its boundary.

Theorem 4.7.3 (Green's)**.** If Q, P are functions of two real variables, with real outputs, with continuous partial derivatives on the domain $\mathcal{D}$, then

$$\int_{\partial R} Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{6}$$

Proof. Assume that $\gamma = \gamma(t)$, for $a \leq t \leq b$. For any t, write $\gamma(t) = x(t) + iy(t)$; the function $f(z) = u(x, y) + iv(x, y)$. Then we can compute the integral:

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\
&= \int_a^b (u(x(t), y(t)) + iv(x(t), y(t))) \cdot (x'(t) + iy'(t))dt \\
&= \int_a^b \left[u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)\right] dt + i\int_a^b \left[u(x(t), y(t))y'(t) - v(x(t), y(t))x'(t)\right] \\
&= \int_\gamma udx - vdy + i\int_\gamma udy + vdx \\
&= \iint_R (-v_x - u_y)dA + i\iint_R (u_x - v_y)dA
\end{aligned}$$

Now since our function is holomorphic, we can use the Cauchy-Riemann equations. This means that both the integrals are 0, as desired. □

4.8 Simply Connected Domains

Definition 4.8.1 (Simply Connected Domain)**.** A *simply connected domain* $\mathcal{D}$ is a domain such that every simple closed curve within it only encloses points of $\mathcal{D}$. Intuitively speaking, it is a domain without holes.

Example. Some examples of simply connected domains are $\mathbb{C}$, $\mathbb{C} \setminus \mathbb{R}^-$. However, $\mathbb{C}^*$ is not a simply connected domain, since any contour which goes around 0 will have points in the interior not in the domain.

The annoying party about the Cauchy-Goursat theorem is that it requires that the interior of the contour is in the domain. However, if a domain is simply connected, then we are by definition guaranteed to have the interior in the domain.

Theorem 4.8.1. Let f be holomorphic on a simply connected domain $\mathcal{D}$. For any closed contour $\gamma \subset \mathcal{D}$,

$$\int_\gamma f(z) = 0.$$

Proof. If γ is a simple closed curve, then we know by definition that the interior of γ is included in $\mathcal{D}$. Therefore, the theorem follows from the first version of the Cauchy-Goursat theorem.

If γ intersects itself at one other point which is not the endpoint, then we can decompose γ into two curves, γ_1 and γ_2 such that they are simple and closed contours. Therefore, the integral can be written as sum over both contours, both of which are 0. This can be extended to any finite number of curves by induction.

In general, infinitely many intersections can be proven, but the proof will not be provided here. □

This version of the theorem requires that $\mathcal{D}$ be simply connected, but it doesn't require that γ be simple. This way, we make no assumptions on the interior of γ.

Example. Let $\mathcal{D} = \{z : |z| < 3\}$. Let $\gamma \subset \mathcal{D}$ be a closed contour. Then

$$\int_\gamma \frac{\sin z}{(z^2+9)^5} = 0.$$

The integrand is defined when $|z| < 3$, and it is holomorphic by the rules of holomorphic functions. In particular, f is holomorphic on $\mathcal{D}$ which is simply connected, so the integral is 0.

What follow are some corollaries of the above theorem:

Corollary 4.8.2. A function f that is holomorphic on a simply connected domain $\mathcal{D}$ must have an antiderivative on $\mathcal{D}$.

Proof. We know that f is continuous on $\mathcal{D}$. Moreover, by the second version of the Cauchy-Goursat theorem, we know that the integral will be 0 for any closed curve γ lying in $\mathcal{D}$. Therfore, statement 3 in the theorem of section 48 is true. Therefore, there is an antiderivative. □

Corollary 4.8.3. Entire functions always posess antiderivatives.

Proof. This follows from the fact that $\mathbb{C}$ is a simply connected domain. □

4.9 Multiply Connected Domains

In a nutshell, a multiply connected domain means not simply connected. Moreover, a holomorphic function f on a multiply connected domain can have an antiderivative or not. For instance, $1/z^3$ and $1/z$ on $\mathbb{C}^*$.

Even though we cannot say that contour integrals are 0, we can still have some independence with respect to the contour for integrals.

Theorem 4.9.1 (Principle of Deformation of Paths)**.** Let γ_1, γ_2 be positively-oriented simple closed contours, where γ_1 is included in the interior of γ_2. If f is a function holomorphic at any point which is on γ_1, γ_2, or between γ_1 and γ_2, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Proof. We would like to apply the Cauchy-Goursat theorem. We must choose our contours wisely before applying it. □

Theorem 4.9.2. Let $\gamma, \gamma_1, \ldots, \gamma_n$ be simple closed contours positively oriented, such that the interiors of $\gamma_1, \ldots, \gamma_n$ are disjoint and all included in the interior of γ.

If f is analytic on all these contours and on the multiply connected domain consisting of the points inside γ and exterior to each γ_k, then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz.$$

Visually, this looks like:
[PICTURE]

4.10 Cauchy Integral Formula

Theorem 4.10.1 (Cauchy Integral Formula)**.** Let f be holomorphic everywhere inside and on a simple closed contour γ, which is positively oriented. If z_0 is any point on the interior of γ, then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz.$$

Proof. We have that z_0 is in the interior of γ which is open. Since it's open, there is an $r > 0$ such that the open ball $B_r z_0$ is included in the interior of γ. Then let γ_r be the contour which is positively-oriented and comprises the boundary of the open ball. By the previous theorem, we know that

$$\int_\gamma = \frac{f(z)}{z - z_0} dz = \int_{\gamma_r} \frac{f(z)}{z - z_0} dz.$$

Now we want to prove that

$$\int_{\gamma_r} \frac{f(z)}{z - z_0} dz = \int_{\gamma_r} \frac{f(z_0)}{z - z_0} dz.$$

Since this integral does not depend on r, we will show this is the case. Let $\varepsilon > 0$. By continuity of f at z_0, there is a $\delta > 0$ such that $0 \le |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$. Then choosing an $r < \delta$, we see that for any z on the contour that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \frac{\varepsilon}{r}.$$

Therefore,

$$\begin{aligned} \left| \int_{\gamma_r} \frac{f(z)}{z - z_0} fz - \int_{\gamma_r} \frac{f(z_0)}{z - z_0} dz \right| &\le \int_{\gamma_r} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \\ &\le \frac{\varepsilon}{r} 2\pi r \\ &= 2\pi\varepsilon. \end{aligned}$$

However, we know that

$$\int_\gamma \frac{f(z)}{z - z_0} dz \int_{\gamma_r} \frac{f(z_0)}{z - z_0} dz = f(z_0) \times 2i\pi$$

from which we obtain our result;

$$\left| \int_\gamma \frac{f(z)}{z - z_0} dz - 2i\pi f(z_0) \right| \le 2\pi\varepsilon$$

for all $\varepsilon > 0$. □

The above is an important theoretical statement, as well as important for calculations. This means that f is completely determined on the interior of γ by its values on γ.

Theorem 4.10.2 (General Cauchy Integral Formula)**.** Let f be holomorphic within and on a positively oriented simple closed contour γ. If z_0 is any point interior to γ, then f is infinitely differentiable at z_0 and

$$f^{(n)}(z_0) = \frac{n!}{2i\pi} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}} dz$$

for any integer $n \geq 0$.

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Now we may consider some consequences of this formula.

Theorem 4.10.3. If a function f is holomorphic at a point z_0, then it is infinitely differentiable in a neighborhood of z_0.

Proof. From the definition of holomorphicity, if f is holomorphic at z_0 means that f is holomorphic in a neighborhood $\{z : |z-z_0| < \varepsilon\}$ of z_0. We do not know that f is holomorphic on the boundary of this circle, but we may choose a circle N of radius $\varepsilon/2$, and let $\gamma = \partial N$. Now all the assumptions of the general Cauchy Integral formula are satisfied (that f is holomorphic on and within γ.) Therefore, by the integral formula, f has infinitely many derivatives on the interior of C, which is a neighborhood of z_0. □

Note that a function may be differentiable at a point, but not holomorphic. It is much stronger, and gives us the remarkable theorem above. The above has the following corollary:

Corollary 4.10.4. If a function $f(z) = u(x,y)+iv(x,y)$ is analytic at z_0, then u and v have partial derivatives at all orders in a neighborhood of z_0.

Proof. This can be proved by induction by using the Cauchy-Riemann equations. □

Theorem 4.10.5 (Cauchy's Inequality)**.** Let $z_0 \in \mathbb{C}$ and $r > 0$, $r \in \mathbb{R}$. Let f be holomorphic function on and within a contour γ_r, which is the positively-oriented circle with radius r. around z_0. Then $|f(z)|$ has a maximal value M_r on γ_r.

Moreover, for any integer $n \geq 0$,

$$\left| |f^{(n)}(z_0) \right| \leq \frac{n! M_r}{r^n}.$$

Proof. We must first prove that there is the maximal value M_r. Its existence is guaranteed, however, by the theorem [THEOREM] since f is continuous on γ_r and since we are on a bounded, closed region.

Then, since f is holomorphic on and within γ_r, by the Cauchy Integral formula, we know that

$$f^{(n)}(z_0) = \frac{n!}{2i\pi} \oint_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Then we can take the modulus, and note that for z on the contour γ_r,

$$\begin{aligned}
\left| f^{(n)}(z_0) \right| &= \frac{n!}{2\pi} \left| \oint_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}} dz \right| \\
&\leq \frac{n!}{2\pi} \oint \frac{M_r}{r^{n+1}} \\
&= \frac{n!}{2\pi} \cdot 2\pi r \frac{M_r}{r^{n+1}} \\
&= \frac{n! M_r}{r^n}
\end{aligned}$$

as desired. □

Example. Assume f is holomorphic on and within the unit circle, and $|f(z)| \leq 1$ on the unit circle. Then $|f(0)| \leq 1$. Then $|f'(0)| \leq 1$. Similarly, $|f^{(n)}(0)| \leq n!$. Note that this only works for the center of the contour, since that is the only point for which the distance to the boundary is 1.

4.11 Liouville's Theorem and the Fundamental Theorem of Algebra

Theorem 4.11.1 (Liouville's Theorem)**.** If a function f is entire and bounded on $\mathbb{C}$, then f is a constant function.

Proof. Let $z_0 \in \mathbb{C}$ and let $r > 0$, $r \in \mathbb{R}$. Then f is holomorphic on and within γ_r, which is the circle centered at z_0 of radius r. Then, we can apply Cauchy's inequality. Let M_r be the maximal value of $|f(z)|$ on γ_r. Then

$$|f'(z_0)| \leq \frac{M_r}{r}$$

However, we also know that f is bounded on $\mathbb{C}$. Then tehre is a constant $M > 0$ such that for any $z \in \mathbb{C}$, $|f(z)| \leq M$. In particular, $M_r \leq M$. Then

$$|f'(z_0)| \leq \frac{M}{r}.$$

Now, the above holds for any r. Therefore, the modulus of the derivative is arbitrarily small, so we must have that $f'(z_0) = 0$. This implies that f is constant on $\mathbb{C}$, from theorem [Theorem] □

This theorem may seem counter-intuitive. For instance, our first insticnt may be to consider sin and cos. However, these are bounded on the real axis and unbounded on the imaginary axis; that is,

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i}.$$

The next question is how do we get the fundamental theorem of algebra from this? First, recall the definitions of a (complex) polynomial and a zero of a polynomial.

Lemma 4.11.2. Let $p(z)$ be a polynomial of degree n. Then there is a constant $r > 0$ such that for any $|z| > r$,

$$|p(z)| \geq \frac{|a_n| r^n}{2}.$$

What this says is basically that the largest term dominates.

Proof. We have that

$$p(z) = a_n z^n + z^n \left(\frac{a_0}{z^n} + \cdots + \frac{a_{n-1}}{z} \right).$$

We can apply the second version of the triangle inequality,

$$\begin{aligned}
|p(z)| \geq |a_n z^n| - |z^n| \left| \frac{a_0}{z^n} + \cdots + \frac{a_{n-1}}{z} \right| & \\
&\geq |a_n||z^n| \frac{|a_0|}{|z^n|} + \cdots + \frac{|a_{n+1}}{|z|} \\
&\geq \frac{|a_0|}{r^n} + \cdots + \frac{|a_{n-1}}{r} \\
&\geq \frac{|a_0| + \cdots + |a_{n-1}|}{r}
\end{aligned}$$

With the choice

$$r \geq \frac{2(|a_0| + \cdots + |a_{n-1}|)}{|a_n|},$$

then

$$\begin{aligned} |p(z)| &\geq |z|^n \left(|a_n| - \frac{|a_n|}{2} \right) \\ &= |z|^n \frac{|a_n|}{2} \\ &\geq r^n \frac{|a_n|}{2} \end{aligned}$$

as desired □

Theorem 4.11.3 (Fundamental Lemma of Algebra)**.** Every polynomial of degree at least one has at least one zero.

Proof. Let p be a polynomial. Assume that p has no zero. Then this will prove that p is constant, which is a proof by contrapositive. Then the function $1/p(z)$ is entire, since $p(z) \neq 0$ for any $z \in \mathbb{C}$. Moreover, $1/p(z)$ is bounded on $\mathbb{C}$. By the above lemma, there is an $r > 0$ such that $|z| > r$ implies that $|p(z)| \geq |a_n| r^n / 2$. This implies that

$$\left| \frac{1}{p(z)} \right| \leq \frac{2}{|a_n| r^n}.$$

Inside the closed disk of radius r, we know that our function is also bounded by Cauchy's inequality.

Therefore, since $1/p(z)$ is entire and bounded, we know that $1/p(z)$ is constant by Liouville's theorem. Therefore, $p(z)$ is constant on $\mathbb{C}$, so it does not have a positive degree. □

Theorem 4.11.4. Let $p(z)$ be a polynomial of degree $n \geq 1$. Then there exist complex constants $c, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that

$$p(z) = c(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n).$$

Remark. Note that the α_i's may appear several times. Therefore, p cannot have more than n distinct zeroes.

5 Sequences and Series

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Definition 5.0.1 (Divergent Series)**.** If a sequence has no limit, we say it divergence.

Theorem 5.0.1. Let (z_n) be a sequence of complex numbers. Write $z_n = x_n + y_n$ for each n. Let $z = x + iy \in \mathbb{C}$. Then

$$\lim_{n\to\infty} = z \iff \begin{cases} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{cases}$$

So we can write that $\lim_{n\to\infty}(x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$ as soon as we know the limit on the left-hand side exists or both limits on the Right Hand Side exist.

Example. Let the sequence

$$z_n = \left(2 + \frac{2}{n}\right) + i\left(1 + \frac{2}{n}\right)$$

then

$$\lim_{n\to\infty} z_n = \left(\lim_{n\to\infty} 2 + \frac{2}{n}\right) + i\left(\lim_{n\to\infty} 1 + \frac{2}{n}\right) = 2 + i.$$

Let $\varepsilon > 0$. Let

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5.1 Taylor Series

You should know that for real functions, we can expand some of them into an infinite Taylor Series, with its derivatives and factorials.

Theorem 5.1.1. Let $z_0 \in \mathbb{C}$, and $r_0 > 0$. Assume a function f is holomorphic on the open disk centered at z_0 with radius r_0. Then for any $|z - z_0| < r_0$, then

$$f(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n. \tag{7}$$

This equation is called the Taylor Series of f at z_0.

How do we apply this theorem? If f is holomorphic on a domain $\mathcal{D}$, let $z_0 \in \mathcal{D}$. Then choose the largest r_0 such that $\{z : |z - z_0| < r_0\}$ is included in $\mathcal{D}$. Then by Taylor's theorem, we know that every point in this maximal disk admits an infinite expansion.

Remark. When $z_0 = 0$, and f is holomorphic on $\{z : |z| < r_0\}$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \tag{8}$$

The proof of Taylor's theorem, though a sweeping statement, can be made concise due to the tools we have already developed. In particular, we know that holomorphicity implies infinite differentiability, and we also have the Cauchy integral formula for disks.

Proof. Let $z \in \mathbb{C}$ such that $|z - z_0| < r_0$. Let $\rho = |z - z_0|$, and let $\rho_0 > 0$ be such that $\rho < \rho_0 < r_0$. Essentially, we create a circle slightly smaller than the larger circle with radius r_0. In particular, we know that f is holomorphic on $B_{\rho_0}(z)$. For $N \geq 0$, let

$$S_K(z) := \sum_{n=0}^{K} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

We want to show that $S_K(z) \to f(z)$. Since f is holomorphic on and within $B_{\rho_0}(z)$, then the Cauchy integral formula gives us

$$f^{(n)}(z_0) = \frac{n!}{2i\pi} \int_{B_{\rho_0}} \frac{f(w)}{(w-z_0)^{n+1}} dw$$

so then

$$\begin{aligned} S_K(z) &= \sum_{n=0}^{K} \frac{1}{2i\pi} \int_{B_{\rho_0}} \frac{f(w)}{(w-z_0)^{n+1}} dw \\ &= \frac{1}{2i\pi} \int_{B_{\rho_0}} \sum_{n=0}^{K} \frac{f(w)}{(w-z_0)^{n+1}} dw \\ &= \frac{1}{2i\pi} \int_{B_{\rho_0}} \left(\frac{f(w)}{w-z_0} \sum_{n=0}^{K} \left(\frac{z-z_0}{w-z_0} \right)^n \right) dw \end{aligned}$$

after applying theorems from the geometric sum, this turns into

$$\frac{1}{2i\pi} \int_{B_{\rho_0}} \left(f(w) \cdot \frac{1 - \left(\frac{z-z_0}{w-z_0}\right)^{K+1}}{w-z} \right) dw$$

Therefore,

$$\begin{aligned} S_K(z) &= \frac{1}{2i\pi} \int_{B_{\rho_0}} \frac{f(w)}{w-z} \left(1 - \frac{z-z_0}{w-z_0} \right) dw \\ &= f(z) - \frac{1}{2i\pi} \int_{B_{\rho_0}} \frac{f(w)}{w-z} \left(\frac{z-z_0}{w-z_0} \right)^{K+1} dw \end{aligned}$$

Now we want to show that $|S_K(z) - f(z)| \to 0$. Therefore,

$$|S_K(z) - f(z)| = \frac{1}{2\pi} \left| \int_{B_{\rho_0}} \frac{f(w)}{w-z} \left(\frac{z-z_0}{w-z_0} \right)^{K+1} dw \right|$$

We know that f is holomorphic, and therefore continuous on B_{ρ_0} which is a closed and bounded region. Therefore, there is an $M > 0$ such that $|f(w)| \le M$ for any w on C_0. Therefore,

$$\begin{aligned} |w-z| \ge |w-z_0| - |z-z_0| &= \rho_0 - \rho \\ \left| \frac{z-z_0}{w-z_0} \right|^{K+1} &= \left(\frac{\rho}{\rho_0} \right)^{K+1} \end{aligned}$$

And therefore, we can say that

$$|S_K(z) - f(z)| \le \frac{1}{2\pi} \times 2\pi\rho_0 \times \frac{M}{\rho_0 - \rho} \left(\frac{\rho}{\rho_0} \right)^{K+1}$$

which goes to 0 as $K \to \infty$. This proves that the series converges and equals □

Now we can consider some examples of this remarkable theorem.

Example. The following functions have these useful expansions:

(1)

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \; |z| < 1$$

(2)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \; z \in \mathbb{C}$$

(3)

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \; z \in \mathbb{C}$$

(4)

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \; z \in \mathbb{C}$$

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[FIRST 2 MINS REWATCH]
There are two ways of finding a function's Taylor expansion. We can plug and chug, using the formula, or we can use expansions of functions we already know.

5.1.1 Negative Powers of $(z - z_0)$

[EXPLANATION] If the function f explodes at z_0, it can still be possible to expand in terms of positive and negative powers of $(z - z_0)$.

Example. Let $f(z) = \frac{e^z - 1}{z^3}$. The limit of this function as $z \to 0$ is infinity, in the complex sense. We have that

$$e^z - 1 = \sum_{k=1}^{\infty} \frac{z^k}{k!}$$

therefore,

$$f(z) = \frac{1}{z^3} \sum_{k=1}^{\infty} \frac{z^k}{k!} = \sum_{n=-2}^{\infty} \frac{z^n}{(n+3)!}$$

which can be written as

$$f(z) = \frac{1}{z^2} + \frac{1}{2z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+3)!}$$

Next, we can try to expand a function in terms of powers.

Example. Let

$$f(z) = z^3 \cosh\left(\frac{1}{z}\right)$$

Then if $z \neq 0$, our taylor expansion formula is

$$\cosh(1/z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{z}\right)^{2k}}{(2k)!}$$

now if $z \neq 0$, we can write

$$\begin{aligned}\cosh(1/z) &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{z}\right)^{2k}}{(2k)!} \\ &= \sum_{k=0}^{\infty} \frac{1}{z^{2k-3}(2k)!} \\ &= z^3 + \frac{z}{2} + \sum_{k=2}^{\infty} \frac{1}{z^{2k-3}(2k)!} \\ &= z^3 + \frac{z}{2} + \sum_{n=0}^{\infty} \frac{1}{z^{2n+1}(2n+4)!}\end{aligned}$$

5.2 Laurent Series

In this section, we will show that such an expansion of positive and negative powers of $(z - z_0)$ is always possible in certain regions, even if f is not holomorphic at z_0.

Theorem 5.2.1 (Laurent's Theorem)**.** Let $z_0 \in \mathbb{C}$ and $0 \leq r_1 < r_2$. Let f be holomorphic in the annular domain

$$\mathcal{D} = \{z : r_1 < |z - z_0| < r_2\}.$$

Let γ be a positively oriented, simple closed contour around z_0 in $\mathcal{D}$. Then for any $z \in \mathcal{D}$, we have that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n} \tag{9}$$

where

$$\begin{aligned}a_n &= \frac{1}{2i\pi} \oint_\gamma \frac{f(w)}{(w - z_0)^{n+1}} dw \\ b_n &= \frac{1}{2i\pi} \oint_\gamma \frac{f(w)}{(w - z_0)^{-n+1}} dw = a_{-n}.\end{aligned}$$

The above is called a Laurent series expansion at z_0.

Proof. If you remember the proof of Taylor's theorem, the idea was to write the partial sum, use the Cauchy integral formula, as well as the geometric sum formula, and then bound the difference and let it go to 0. The proof of this theorem relies on considering two positively oriented, cicrcular contours γ_1 and γ_2 of radii ρ_1 and ρ_2 which are arbitrarily close to the circles with radii r_1 and r_2. let γ' be a simple closed contour around z lying between γ_1 and γ_2. Then

$$\oint_{\gamma_2} \frac{f(w)}{(w - z)} dw = \int_{\gamma'} \frac{f(w)}{w - z} dw + \int_{\gamma_1} \frac{f(w)}{w - z} dw$$

which is a fact we proved a few sections ago about multiply connected domains. By the cauchy integral formula,

$$\oint_{\gamma'} \frac{f(w)}{w - z} dw = 2i\pi f(z)$$

since f is analytic in the annulus. Therefore,

$$f(z) = \frac{1}{2i\pi} \int_{\gamma_2} \frac{f(w)}{w - z} dw - \frac{1}{2i\pi} \int_{\gamma_1} \frac{f(w)}{w - z} dw.$$

Then we have the following two claims we can make:

(1) The first term is that the integral around γ_2 is equal to

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

(2) The second term, the integral around γ_1, is

$$-\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$$

The proof of both of these claims is proved in a really similar way to Taylor's theorem. $\square$

Remark. Since $b_n = a_{-n}$ we may write the Laurent series as

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n.$$

Remark. If f is holomorphic throughout the disk of radius r_2, then this theorem gives the Taylor series of f at z_0, from the genearl Cauchy integral formula; the b_n's mean that we get the product of two holomorphic functions so the integral is simply 0 by the Cauchy-Goursat theorem.

Generlly speaking, we do not use the integral representation of a_n and b_n to find the Laurent series. Rather, we use the usual Taylor expansion as in the previous section.

Laurent's theorem can help to predict the domain(s) in which an expansion will be valid.

Example. Let

$$f(z) = \frac{1}{z(z^2+1)} = \frac{1}{z(z-i)(z+i)}$$

and the goal is to get an expansion in powers of z, or where $z_0 = 0$. Now we can consider the domain $\mathcal{D}_1$ where $0 < |z| < 1$. Our second domain can be everywhere outside of this disk of radius 1, or $\mathcal{D}_2$ given by $1 < |z| < \infty$. These are two annuli where we can expect a Laurent series expansion to hold. On $\mathcal{D}_1$, we can use the familiar formula

$$\begin{aligned}\frac{1}{1+z^2} &= \frac{1}{1-(-z^2)} \\ &= \sum_{n=0}^{\infty}(-z^2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n z^{2n} \\ &= \frac{1}{z} + \sum_{k=0}^{\infty}(-1)^{k+1} z^{2k+1}.\end{aligned}$$

For the other domain $\mathcal{D}_2$,

$$\begin{aligned}\frac{1}{1+z^2} &= \frac{1}{z^2} \times \frac{1}{1-(-1/z^n)} \\ &= \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{-1}{z^2}\right)^n\end{aligned}$$

Therefore, multiplying everything by $1/z$, we get

$$\begin{aligned} f(z) &= \frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{-1}{z^2}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n z^{-2n-3}. \end{aligned}$$

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Therefore, $(a_n r^n)_{n\geq 0}$ is bounded if and only if $r \leq \frac{1}{2}$.

etc

Remark. If $0 \leq q < 1$ and $\alpha \in \mathbb{R}$,then $n^\alpha q^n \to 0$. Moreover, if $q > 1$ and $\alpha \in \mathbb{R}$, then $n^\alpha q^n \to \infty$ as $n \to \infty$. Lastly, if $q > 0$, then $n!q^n \to \infty$ and $\frac{q^n}{n!} \to 0$ as $n \to \infty$.

Theorem 5.2.2. Let

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

be a power series with radius of convergence R. If $|z-z_0| < R$, then the series is absolutely convergent; if $|z-z_0| > R$, then the series diverges.

Proof. Let $r = |z-z_0| < R$. Fix some ρ such that $r < \rho < R$. Since $\rho < R$, we know that $(a_n\rho^n)_{n\geq 0}$ is bounded. So there is an $M > 0$ such that

$$|a_n\rho^n| \leq M$$

for any n. Therefore,

$$\begin{aligned} |a_n(z-z_0)|^n &= |a_n|\, r^n \\ &= |a_n\rho^n| \times \left(\frac{r}{\rho}\right)^n \\ &\leq M \times \left(\frac{r}{\rho}\right)^n. \end{aligned}$$

Since $0 \leq r/\rho < 1$, we know that

$$\sum_{n=0}^{\infty} M \times \left(\frac{r}{\rho}\right)^n$$

converges. Therefore, our original sum

$$\sum_{n=0}^{\infty} a_n(z-z_0)$$

converges absolutely.

Next, let $r = |z-z_0| > R$. Since $r > R$, we know that $(a_n r^n)_{n\geq 0}$ is not bounded. Therefore, $(a_n(z-z_0)^n)_{n\geq 0}$ is not bounded because the moduli of both are the same. Therefore,

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

diverges.

□

Exactly on the circle of radius R, the series can be convergent or not. Indeed, it may even depend on the point z on the circle.

Not covered: Theorem 2 in section 69, which introduces uniform convergence. Skipped is section 70 which covers a tool for proofs, weaker than the following results.

5.3 Integration and Differentiation of Power Series

Theorem 5.3.1 (Uniformity of Power Series)**.** Let $f(z) = \sum a_n(z-z_0)^n$ be a power series with radius of convergence R. Let $\mathcal{D} = \{z : |z-z_0| < R\}$. Then the following is true:

(1) f is analytic on $\mathcal{D}$ and $f'(z) = \sum_{n=1}^{\infty} a_n n(z-z_0)^{n-1}$ for any $z \in \mathcal{D}$.

(2) If $\gamma \subset \mathcal{D}$, and g is continuous on γ, then

$$\int_\gamma g(z)f(z)dz = \sum_{n=0}^{\infty} a_n \int_\gamma g(z)(z-z_0)^n dz.$$

Proof. Relies on uniform convergence. □

In general, as known from analysis, one cannot switch the order between differentiation and integration and an infinite sum.

Example. We have seen that if $|z-1| < 1$, then

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

For any $|z-1| < 1$, we can differentiate and multiply by -1 to get

$$\frac{1}{z^2} = \sum_{k=0}^{\infty}(-1)^k(k+1)(z-1)^k.$$

Moreover, we can use power series to prove holomorphicity.

Example. Consider

$$f(z) = \begin{cases} \dfrac{e^z-1}{z} & z \neq 0 \\ \\ 1 & z = 0. \end{cases}$$

We can summarily prove that f is holomorphic on $\mathbb{C}$. The first approach would be to use differentiation rules to prove that f is holomorphic on $\mathbb{C}^*$, and then treat f is differentiable at 0 using the definition. Alternatively, we can write f as a power series. For $z \neq 0$,

$$f(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}$$

which also works for $z = 0$. Therefore, since the radius of convergence of this power series is infinite, we have holomorphicity on $\mathbb{C}$.

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So far, we have seen that if f is analytic on a disk, it will be equal to its taylor series. If we are on an annular domain, then we can represent f by a Laurent series. If f is a power series, then f is analytic on the disk of convergence. We can now write a similar result for Laurent series

Theorem 5.3.2 (Uniformity of Laurent Series)**.** Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$$

be a series converging at any point z in an annular domain $\mathcal{D}$ centered at z_0. Then:

(1) f is analytic on $\mathcal{D}$ and

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1} + \sum_{n=1}^{\infty} -nb_n(z-z_0)^{-n-1}$$

(2) if γ is a contour included in $\mathcal{D}$ and g is a continuous function on γ, then

$$\int_\gamma g(z)f(z)dz = \sum_{n=0}^{\infty} a_n \int g(z)(z-z_0)^n dz + \sum_{n=1}^{\infty} b_n \int g(z)(z-z_0)^n dz$$

Proof. This can be shown via the above theorem and a change of variables in contour integrals. □

5.4 Uniqueness of Series Representations

In all our examples thus far, we have assumed that each function had a unique power series representation. This is indeed true, but it is a result that must first be proven.

Theorem 5.4.1. Let $z_0 \in \mathbb{C}$ and $r > 0$. Let f be a function. Assume that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for any $|z - z_0| < r$. This series is *the* Taylor series of f at z_0. That is,

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

for any $k \geq 0$.

Proof.

(1) (Case $k = 0$). Then we have that

$$f(z_0) = \sum_{n=0}^{\infty} a_n 0^n = a_0.$$

Then we have that

$$a_0 = f(z_0) = \frac{f^{(0)}(z_0)}{0!}$$

(2) (Case $k = 1$). We have not yet assumed that f is analytic, but note that the power series $\sum a_n(z-z_0)^n$ is converging at any point on the disk $\{z : |z - z_0| < r\}$ so its radius of convergence $\geq r$. Therefore, $f(z)$ is holomorphic on this disk, and we have that

$$f'(z) = \sum_{n=1}^{\infty} a_n n(z-z_0)^{n-1}$$

so $f'(z_0) = a_1 \cdot 1 \cdot 0^0 = a_1$ so therefore

$$a_1 = \frac{f'(z)}{1!}$$

(3) (Inductive Hypothesis). By induction, we can prove that

$$f^{(k)}(z) = \sum_{n=k}^{\infty} a_n n(n-1)\cdots(n-(k-1))(z-z_0)^{n-k}$$

for any $|z - z_0| < r$. So then

$$f^{(k)}(z_0) = a_k k(k-1)(k-2)\cdots(k-(k-1)) \times 0^0$$

which gives us

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

as desired.

$\square$

Let us now write the same theorem for Laurent series:

Theorem 5.4.2 (Uniqueness of Laurent Series Expansion)**.** Let $z_0 \in \mathbb{C}$ and $\mathcal{D}$ be an annular domain centered at z_0. Let f be a function. Assume that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$$

Then we have

$$\begin{aligned} a_k &= \frac{1}{2i\pi}\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}dz \\ b_k &= \frac{1}{2i\pi}\int_\gamma \frac{f(z)}{(z-z_0)^{-k+1}}dz \end{aligned}$$

for any simple closed contour γ around z_0 included in $\mathcal{D}$. Note that in particular this means that the representation does not depend on the γ we choose, by the principle of deformation of paths.

First, we must prove a lemma:

Lemma 5.4.3. We claim that

$$\frac{1}{2i\pi}\int_\gamma (z-z_0)^\ell dz = \begin{cases} 0 & \ell \neq -1 \\ 1 & \ell = -1. \end{cases}$$

Proof.

1. If $\ell \geq 0$, we can apply Cauchy-Goursat.
2. If $\ell < 0$, we can apply the Cauchy integral formula with $g(z) = 1$:

$$\begin{aligned} \frac{1}{2i\pi}\int_\gamma (z-z_0)^\ell dz &= \frac{1}{2i\pi}\int_\gamma \frac{g(z)}{(z-z_0)^{(-\ell-1)+1}}dz \\ &= \frac{g^{(-\ell-1)}(z_0)}{(-\ell-1)!} \\ &= \begin{cases} 0 & \ell \neq -1 \\ 1 & \ell = -1. \end{cases} \end{aligned}$$

□

Now we can prove the theorem:

Proof. First, note that from theorem THEOREM, we know that f is holomorphic on the domain. For any $k \in \mathbb{Z}$, we can integrate term by term so thus

$$\frac{1}{2i\pi}\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}dz = \sum_{n=0}^{\infty} a_n \frac{1}{2i\pi}\int_\gamma \frac{(z-z_0)^n}{(z-z_0)^{k+1}}dz + \sum_{n=1}^{\infty} b_n \frac{1}{2i\pi}\int_\gamma \frac{(z-z_0)^{-n}}{(z-z_0)^{k+1}}dz$$

Now from the previous lemma, we can replace $\ell = n-k-1$. If $k \geq 0$ then the integral

$$\frac{1}{2i\pi}\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}dz$$

is just a_k. If $k < 0$ then the integral is b_{-k}. This proves the theorem. □

5.5 Uniquely Determined Holomorphic Functions

Now we can talk about *the* Taylor or Laurent series on a given domain. What comes next is an important result we alluded to in earlier sections: if a holomorphic function's values are known on a large enough set (e.g., a line segment), then this is enough to completely determine the function elsewhere on a domain. Lemma (theroem 3 sec 82)

Lemma 5.5.1. Let $z_0 \in \mathbb{C}$, $r > 0$, and let f be a function holomorphic on

$$\mathcal{D} = \{z : |z - z_0| < r\}.$$

Assume that $f = 0$ on a line segment of positive length containing z_0. Then $f = 0$ on $\mathcal{D}$.

Proof.

(1) If f is holomorphic and $f = 0$ on L, then $f' = 0$ on L. This is true because We can consider h approaching 0 such that $z + h$ is on L. Then $f(z + h) = f(z) = 0$. Therefore, $f'(z) = 0$.

(2) By induction, using step (1), then we can deduce that $f^{(n)} = 0$ on L for all $n \geq 0$.

(3) We have that f is holomorphic on the disk $\mathcal{D}$. By Taylor's theorem, we know that there is a power series expansion on the disk $\mathcal{D}$ given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

However, we know that from step (2), the derivatives are all identically 0. Therefore, $f(z) = 0$ for any $z \in \mathcal{D}$.

□

Now instead of disks, we want a theorem which is true for any domain $\mathcal{D}$, and we want this to be true for a function g which is also holomorphic.

Theorem 5.5.2. Let $\mathcal{D}$ be a domain, and let f, g be holomorphic on $\mathcal{D}$. If $f = g$ on a line segment L included in $\mathcal{D}$, then $f = g$ on all of $\mathcal{D}$.

Proof. If the theorem is true with $g = 0$, then for a general g, we can consider the function $(f - g) = 0$ on the line segment, which is holomorphic.

Now we can consider the case with $g = 0$. For any other poin $w \in \mathcal{D}$, we know that there is a polygonal line connecting z and w. Moreover, since $\mathcal{D}$ is open by definition, we can chain multiple overlapping disks which are completely included in $\mathcal{D}$. Then we can propagate the information all the way over to w. $\square$

Example. We know that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ for $\theta \in \mathbb{R}$. Then this can be expanded to include all complex numbers. Let $f(z) = \sin(2z)$ and $g(z) = 2\sin(z)\cos(z)$. Then without any computations, we know that this is true for all of $\mathbb{C}$.

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6 Residues and Poles

With the new theory developed and tools from series, we will be able to evaluate problematic integrals such as

$$\oint_\gamma \frac{1}{\sin(z)}dz, \oint_\gamma \sin\left(\frac{1}{z}\right)dz$$

for which the Cauchy integral formula cannot be applied due to the singularities on or in the interior of the contour γ. We will define a class of these singularities in order to learn how to deal with them.

6.1 Isolated Singular Points

Recall that a singularity or a singular point z_0 of a function f is a point such that:

(1) f is not holomorphic at z_0, and

(2) f is holomorphic at some point in each neigborhood of z_0.

For instance, the set $\mathcal{D}$ of points where f is holomorphic is always an open set. This is a consequence of the definition of holomorphicity. Moreover, the union of open sets is always open.

Moreover, the singularities of f are the points on the boundary of $\mathcal{D}$, assuming there is a boundary. In general, if $\mathcal{D} = \mathbb{C}$, then we consider it not to have a boundary. We will, however, take time to discuss the point at infinity. Sometimes, it may make sense to consider the point ∞ as the boundary of the entire claim.

Definition 6.1.1 (Isolated Singularity)**.** Let f be a function. A point z_0 is an isolated singularity of f if

(1) f is not holomorphic at z_0 and

(2) There is a deleted neighborhood of z_0 on which f is analytic.

Note that this definition is much stronger. That is,

$$\text{isolated singularity} \;\Rightarrow\; \text{singularity}.$$

We can think of these as "pinpricks" in the set $\mathcal{D}$ where f is holomorphic. Note that the boundary, which consists of singularities, do not have deleted regions such that f is holomorphic.

Example. The function

$$f(z) = \frac{z}{(z-1)(z^2+4)}$$

is holomorphic on $\mathbb{C} \setminus \{1, 2i, -2i\}$.

Example. Consider the function $f(z) = \mathrm{Log}(z)$. This is holomorphic on $\mathbb{C} \setminus \mathbb{R}^-$. Note that since this is a line, no point on that line can be considered an isolated singularity, since any deleted neighborhood will contain singularities.

Example. Lastly, consider the surprising example:

$$f(z) = \frac{1}{\sin(\pi/z)}.$$

This function is not defined at $z = 0$ as well as the zeroes of $\sin(\pi/z)$ so when $z = 1/n$ where $n \in \mathbb{Z} \setminus \{0\}$. Therefore, we can say that f is defined on

$$\mathbb{C} \setminus \left\{0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \dots\right\}$$

and it is holomorphic on this set by the rueles of differentiation. Each of the points 0 and $1/n$ are singularities. However, the only singularity which is not isolated is 0, and the rest are isolated. Around each of the singularities of the type $\pm 1/n$, we can construct a deleted neighborhood of radius $1/n + 1$ where we do not encounter singularities. However, by the archimedian property of the integers, we cannot do something similar for 0.

Definition 6.1.2. If for some $r > 0$ the function f is holomorphic on the set $\{z : |z| > R\}$, then ∞ is said to be an isolated singularity of f.

6.2 Residues

Let z_0 be an isolated singularity of a function f. Then by definition, there is an $r > 0$ such that f is holomorphic on the annular domain given by

$$\{z : 0 < |z - z_0| < r\}.$$

By Laurent's theorem, we know that f has an expansion on this radius:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},\ 0 < |z - z_0| < r$$

and particualrly, a_n and b_n are uniquely defined and given by contour integrals. We know that

$$b_1 = \frac{1}{2i\pi} \oint_\gamma f(z) dz$$

for any positively-oriented, simple closed contour γ around z_0 in our set.

Definition 6.2.1. The coefficient b_1 in the Laurent expansion is called the *residue* of f at an isolated singularity z_0. It is denoted by

$$\operatorname{Res}(f, z_0).$$

Our goal is to find $\oint_\gamma f(z)dz$ where γ is a positively-oriented simple closed contour around the isolated singularity z_0 included in our set.

The method can be sketched out in rough detail:

(1) We can first write $f(z)$ as a Laurent series at z_0 on $\mathcal{D}$ by any means, but not by using the explicit formula of the integral.

(2) In particular, we need to find the coefficent b_1 appearing in the term

$$\frac{b_1}{z - z_0}.$$

There is no need to find the whole Laurent series.

(3) From this, we can conclude that

$$\oint_\gamma f(z)dz = 2i\pi b_1 = 2i\pi \operatorname{Res}(f, z_0)$$

Now we can tackle our examples.

Example. Consider the function

$$f(z) = \sin\left(\frac{1}{z}\right)$$

We cannot apply Cauchy-Goursat since there is a singularity in our domain. For any $z \neq 0$,

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{(1/z)^{2k+1}}{(2k+1)!} = \frac{1}{z} + \dots$$

Therefore,

$$\operatorname{Res}(f, 0) = 1$$

so

$$\oint_\gamma f(z)dz = 2i\pi \operatorname{Res}(f, 0) = 2i\pi.$$

Example. Let

$$f(z) = \frac{e^{z^2} - 1}{z^7}.$$

This is holomorphic on $\mathbb{C}^*$. Therefore, 0 is an isolated singularity. This could be treated by the Cauchy integral formula, but this entails taking a 6th-order derivative which is complicated. Therefore, we elect to expand f as a Laurent series around z_0:

$$\begin{aligned}
e^{z^2} &= \sum_{k=0}^{\infty} \frac{z^{2k}}{k!} \\
e^{z^2} - 1 &= \sum_{k=1}^{\infty} \frac{z^{2k}}{k!} \\
\frac{e^{z^2} - 1}{z^7} &= \sum_{k=1}^{\infty} \frac{z^{2k-7}}{k!} = \dots + \frac{1}{z \times 3!} + \dots
\end{aligned}$$

and so considering the first negative coefficient, we see that occurs when $k = 3$. Thus

$$\int_\gamma f(z)dz = 2i\pi \times \frac{1}{6} = \frac{i\pi}{3}.$$

As should be clear to you, using Residues can greatly simplify the calculation of certain integrals.

6.3 Cauchy's Residue Theorem

There is a more general theorem which encapsulates what we have seen so far.

Theorem 6.3.1 (Cauchy's Residue Theorem)**.** Let γ be a positively-oriented simple closed contour. If f is holomorphic on and within γ, except at finitely many singular points (hence isolated singular points) on the interior of γ, then

$$\oint_\gamma f(z)dz = 2i\pi \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

Proof. Since there are finitely many singular points in the interior, these are isolated singularities. We can choose small enough positively-oriented circular contours $\gamma_1, \ldots, \gamma_n$ around $z_1, \ldots, z_n$ such that f is holomorphic on and within γ_k except at z_k, for all $1 \leq k \leq n$. Since f is holomorphic on the cotours, then by theorem [THEOREM], we know that the integral is equal to

$$\oint_\gamma f(z)dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz.$$

But for each k, we know that

$$\int_{\gamma_k} f(z)dz = 2i\pi \operatorname{Res}(f, z_k)$$

since f is holomorphic on a deleted neighborhood of z_k containing γ_k. This completes the proof. □

Note that there are no disk constraints for this theorem,

Example. Consider the function

$$f(z) = \frac{e^z - 1 - z}{z^3(z-1)}.$$

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can best work w neighborhoods whne we dont need to specify ε. For our purposes, a neighborhood is circular. However, from a topological viewpoint, we can construct a topology from balls or from open sets. Therefore, we really do not lose any properties.

5/4 Most of the time, the way to find Laurent Series is to use known expansions to compute them. In particular, if we want to expand the series around 0 (in order to get the answer in terms of z and its powers), then

Example. Consider the example

$$f(z) = \frac{1}{z - z_0}.$$

Then

$$\frac{1}{z - z_0} = -\frac{1}{z_0} \times \frac{1}{1 - (z/z_0)} = \frac{1}{z} \times \frac{1}{1 - (z_0/z)}$$

Then we know that we can use the geometric series, *provided that* $|z/z_0| < 1$ for the first one and $|z_0/z| < 1$ for the second. Then we get

$$-\frac{1}{z_0} \sum \left(\frac{z}{z_0}\right)^n \quad \left|\frac{z}{z_0}\right| < 1$$
$$\frac{1}{z} \sum \left(\frac{z_0}{z}\right)^n \quad \left|\frac{z_0}{z}\right| < 1$$

For a square, we can do a similar trick and get

$$\frac{1}{(z - z_0)^2} = \frac{1}{z_0^2} \times \frac{1}{(1 - z/z_0)^2} = \frac{1}{z^2} \times \frac{1}{(1 - z_0/z)^2}$$

and do the expansion

$$\frac{1}{(1-w)^2} = \sum (n+1) w^n$$

where we obtain this by differentiating the geometric series, and $|w| < 1$.

If we want to expand the above around 1, then

$$\frac{1}{z - z_0} = \frac{1}{(z-1) - (z_0 - 1)}$$

and do the same as before with $z - 1$ instead of z.

Concerning power series, keep in mind the theorem of section 69:
Let

$$f(z) = \sum a_n (z - z_0)^n$$

and assume that the series converges for a given point z_1. We do not know the radius is convergent, since we don't know the a_n. However, since we know that the radius of convergence R is a disk, then everywhere outside this disk, $|z - z_0| > R$ then the series diverges. Therefore, $|z_1 - z_0| \leq R$. This is a lower bound for the radius of convergence. Moreover, the series is converging for any z such that $|z - z_0| < |z - z_1|$.

Note the strict inequality; we cannot say anything about the boundary, since some series do not converge everywhere on the boundary:

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

since $R = 1$. The series diverges at 1 but converges at -1.

6.4 The Residue at Infinity

Let f be a function such that ∞ is an isolated singularity. What we mean by this is that there is some $R_1 > 0$ such that f is holomorphic on

$$\mathcal{D} = \{z : |z| > R_1\}.$$

Note that this domain is annular centered at 0. Therefore, we can apply Laurent's theorem. We get a unique expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

for $z \in \mathcal{D}$.

Definition 6.4.1. The residue of f at ∞ is defined at

$$\operatorname{Res}(f, \infty) = -b_1.$$

The point of the above definition is to aid in the computation of integrals.

Theorem 6.4.1. Let γ be a positively oriented simple closed contour. Let f be holomorphic on and *outside* γ. Then

$$\oint_\gamma f(z)dz = -2i\pi \operatorname{Res}(f, \infty) = 2i\pi \operatorname{Res}\left(\frac{1}{z^2} f(1/z), 0\right). \tag{10}$$

Proof. We can look at a circle of radius r_1 centered around 0 such that the contour γ is completely contained in the circle. We know that f is holomorphic outside the contour, so in particular it is holomorphic outsidfe of this disk. Then we can expand $f(z)$ outside of this circle:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

for $|z| > r_1$ where

$$b_1 = \frac{1}{2i\pi} \int_{\gamma'} f(z)dz$$

for any contour γ' contained in $\{z : |z| > r_1\}$. In particular,

$$\frac{1}{2i\pi}\int_{\gamma'} f(z)dz = -2i\pi \operatorname{Res}(f,\infty).$$

By the principle of deformation of paths,

$$\int_{\gamma} f(z)dz = \int_{\gamma'} f(z)dz.$$

since f is holomorphic on γ, γ', and everywhere between.

To prove the second part, we will prove that

$$\operatorname{Res}(f,\infty) = -\operatorname{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right).$$

If $|z| < 1/r_1$, then $|1/z| > r_1$, so we can expand the Laurent series, using $1/z$ instead of z:

$$\begin{aligned}\frac{1}{z^2}f\left(\frac{1}{z}\right) &= \frac{1}{z^2}\left(\sum_{n=0}^{\infty} a_n\left(\frac{1}{z}\right)^n + \sum_{n=1}^{\infty} b_n\left(\frac{1}{z}\right)^{-n}\right)\\ &= \sum_{n=0}^{\infty} a_n z^{-2-n} + \sum_{n=1}^{\infty} b_n z^{-2+n}\\ &= \cdots + \frac{a_0}{z^2} + \frac{b_1}{z} + b_2 + \dots\end{aligned}$$

which completes the proof. □

Most of the time, we use the second conclusion of this thorem to simplify computations. One can use Cauchy's residue theorem, but if there are multiple singularities in the interior of a contour, we may opt to do this method.

Example. Consider the function

$$f(z) = \frac{z^4}{(z^4-1)(z-2)}.$$

Note that f is holomorphic on $\mathbb{C} \setminus \{1, -1, -, -i, 2\}$. Therefore, if our simple closed contour γ is outside of all the singularities, then

$$\int_{\gamma} f(z)dz = 2i\pi \operatorname{Res}\left(\frac{1}{z^2}f(\frac{1}{z}), 0\right).$$

Then

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z^2} \times \frac{(1/z)^4}{((1/z)^4-1)(1/z-2)} = \frac{1}{z} \times \frac{1}{(1-z^4)(1-2z)}$$

For this function, there are singularities at 0 due to the first multiplicative term. If we label the second term $g(z)$, then $g(z)$ is holomorphic on the disk centered at 0 of radius 1/2. By Taylor's theorem, we can write $g(z)$ as a series. Therefore,

$$\frac{1}{z^2}f(\frac{1}{z}) = \frac{1}{z}g(z) = \frac{a_0}{z} + a_1 + a_2 z + \dots \; |z| < \frac{1}{2}$$

Clearly,

$$\operatorname{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right) = a_0 = g(0) = 1.$$

Therefore,

$$\oint_{\gamma} f(z)dz = 2i\pi.$$

This method we used for finding the residue will be generalized in later section. Moreover, we should just focus on finding the term with z in the denominator.

6.5 Classifying Isolated Singularities

Let f be a function with an isolated singularity at z_0. Then, recalling the definition, we know that there is a ball around z_0 on which f is holomorphic except at z_0. By Laurent's theorem, this function has an expansion. There are 3 cases for this:

Definition 6.5.1 (Removable Singularity, Essential Singularity, Pole)**.** Let f be a function with an isolated singularity at z_0. Then

(1) If all the b_n's are zero, then z_0 is said to be a removable singularity for f. By setting $f(z_0) = a_0$, this defines a function which is holomorphic at z_0. This effectively "removes" the singularity, since the function seems to have an artificial isolated point. In this case,
$$\operatorname{Res}(f, z_0) = 0.$$

(2) If infinitely many b_n are nonzero, then z_0 is said to be an *essential* singularity.

(3) If finitely many b_n are nonzero, then there is an integer $m \geq 1$ such that $b_n = 0$ for $n > m$. Then we say that z_0 is a *pole* of order m. That is,
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}.$$

Let us now consider a few examples of these singularities.

Example. Let
$$f(z) = \frac{1 - \cos(z)}{z^2}$$
for $z \neq 0$. Moreover, f has an isolated singularity at z_0. For $z \neq 0$,
$$\begin{aligned} f(z) &= \frac{1}{z^2}\left(1 - \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}\right) \\ &= \frac{1}{z^2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2n}}{(2n)!} \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^{2(n-1)}}{(2n)!} \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k+2)!} \end{aligned}$$

Notice that we only have non-negative powers of z. Therefore, it is a removable singularity. Then set $f(0) = 1/2$, and sp f is holomorphic at 0.

Example. Suppose
$$f(z) = \cos(1/z)$$
for $z \neq 0$. Then
$$f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{-2n}}{(2n)!}$$

and

$$b_{2n} = \frac{(-1)^n}{(2n)!} \neq 0$$

but $b_{2n+1} = 0$.

Example. Let

$$f(z) = \frac{1}{z^2(2-z)}.$$

We see that f has 2 singularities, one at 0 and the other at 2. At $z_0 = 0$,

$$\begin{aligned} f(z) &= \frac{1}{2z^2} \times \frac{1}{1-z/2} \\ &= \frac{1}{2z^2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \quad |z| < 2 \\ &= \frac{1}{z^2} + \frac{1}{4z} + \frac{1}{8} + \frac{z}{16} + \dots \end{aligned}$$

so 0 is a pole of order 2, and $\operatorname{Res}(f, 0) = 1/4$. We can do simular computations for $z_0 = 2$, and we find that it is a pole of order 1 whose residue is [RESIDUE].

[END OF LECTURE] If you have a function holomorphic except at finitely many point, then in particular we have that infinity is an isolated singularity, since we can enclose all of them.

For a given contour, by deformation of paths, we can compute small circles around the singularities on the outside of our contour. Then let γ enclose some of the singularities, and γ_1 enclose z_1 and γ_2 enclose z_2.

$$\oint_{\gamma'} = \int_{\gamma} + \int_{\gamma_1} + \int_{\gamma_2}$$

And so

$$\int_{\gamma} f(z)dz = -2i\pi \left(\operatorname{Res}(f, \infty) + \operatorname{Res}(f, z_1) + \operatorname{Res}(f, z_2)\right)$$

So note that singularities on the outside, we have negative contributions; hence, we considered ∞ to be a point exterior to all of $\mathbb{C}$.

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6.6 Residues at Poles

[FIRST 8 MINS]

Theorem 6.6.1. Let z_0 be an isolated singularity of a function f. Let $m \geq 1$ be an integer. Then the following are equivalent:

1. There is a function ϕ, analytic and nonzero at z_0, such that

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

   for z in a deleted neighborhood at z_0.

2. The point z_0 is a pole of order m.

Moreover, if (1) and (2) are true, then

$$\operatorname{Res}(f, z_0) = \frac{\phi^{(m-1)(z_0)}}{(m-1)!}$$

Proof. 1. Suppose there exists such a function ϕ satisfying property (1). Let N be a neighborhood of z_0. Since f is holomorphic on N, then we can select ϕ such that

$$\frac{\phi(z)}{(z-z_0)^m}$$

and $\phi(z_0) \neq 0$. By Taylor's theorem,

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for $z \in N$. Then, if $z \neq z_0$,

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} = \sum_{k=-m}^{\infty} a_{k+m}(z-z_0)^k.$$

which is

$$\frac{a_0}{(z-z_0)^m} + \cdots + \frac{a_{m-1}}{(z-z_0)} + \sum_{k=0}^{\infty} a_{k+m}(z-z_0)^k.$$

Since $a_0 = \phi(z_0)$, which is nonzero, then z_0 is a pole of order m for f. Moreover, we want the residue, which is

$$\operatorname{Res}(f, z_0) = a_{m-1} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

2. Exercise.

□

How do we use this? If f is a function with an isolated singularity at z_0. Then we could attempt to classify the type of singularity, as in the previous section. One way to do this is to expand f as a power series. We can show that z_0 is a pole, and find its order. We write f as

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

for some $m \geq 1$ and some ϕ. The method is to check that ϕ is holomorphic and nonzero at z_0, and then by the theorem, conclude that z_0 is a pole of order m.

The second typical question would be to find the residue of f at this point. The method for this is very similar; we do the first two steps, but then conclude by the theorem that

$$\operatorname{Res}(f, z_0) = \frac{\phi^{(m-1)(z_0)}}{(m-1)!}.$$

Remark. Let γ be a POSCC simple closed contour around z_0. Assume that f is holomorphic on and within γ, except at z_0. Then by Cauchy's residue theorem,

$$\oint_\gamma f(z)dz = 2i\pi \operatorname{Res}(f, z_0).$$

If one can write

$$\operatorname{Res}(f, z_0) = \frac{\phi(z)}{(z-z_0)^m}$$

for some $m \geq 1$ and ϕ holomorphic and nonzero at z_0, then the equality becomes

$$\oint_\gamma \frac{\phi(z)}{(z-z_0)^m} dz = 2i\pi \frac{\phi^{(m-1)(z_0)}}{(m-1)!}$$

which is just the Cauchy integral formula.

In this sense, Residues are stictly stronger than the Cauchy integral formula, in terms of efficiency. Using Cauchy's reidue theorem and the theorem above accoplishes this for a larger set of problems.

Thus, we can see some instances of the above two questions.

Example. Let
$$f(z) = \frac{z-i}{(z+1)(z-2)^2}.$$
We want to find the singularities of f, and find the integral of f centered at 0 on a circle of radius 3. Suppose f is holomorphic on $\mathbb{C} \setminus \{-1, 2\}$. At $z_0 = -1$,
$$f(z) = \frac{\phi(z)}{z+1}$$
where
$$\phi(z) = \frac{z-i}{(z-2)^2}$$
and $\phi(-1) = -\frac{1}{9}(1+i) \neq 0$. Thus, we conclude that -1 is a pole of order 1, and the residue is
$$\operatorname{Res}(f, -1) = \phi(-1) = -\frac{1}{9}(1+i).$$
At $z_0 = 2$, then we write
$$f(z) = \frac{\psi(z)}{(z-2)^2},$$
where
$$\psi(z) = \frac{z-i}{z+1}$$
and $\psi(2) \neq 0$. Thus, 2 is a pole of order 2, and
$$\operatorname{Res}(f, 2) = \frac{\psi^{(1)}(2)}{1!} = \frac{1+i}{(z+1)^2} = \frac{1}{9}(1+i)$$
By Cauchy's residue theorem, the integral around γ is
$$\oint_\gamma f(z)dz = 2i\pi\left(\operatorname{Res}(f, -1) + \operatorname{Res}(f, 2)\right) = 0.$$

We want to know when we should use this technique or to simply expand the function. The following is an illustration when the original technique is more effective.

Example. Let
$$f(z) = \frac{e^z - 1}{z^5}.$$
In order to find the integral around 0, we can first try to use the above method, with $\phi(z) = e^z - 1$. However, as you can see, this doesn't work since $\phi(0) = 0$. Thus, this function is not a pole of order 5, so it is compensating for some of the irregularity at 0.

Our second try can be $\phi(z) = (e^z - 1)/z$, which is 1 at $\phi(0)$. Moreover, ϕ is holomorphic, since it can be written as a power series, and so the residue is given by
$$\operatorname{Res}(f, 0) = \frac{\phi^{(3)}(0)}{3!}$$

The efficient way to find $\phi^{(3)}(0)$ is to use the power series expansion of $\phi(0)$. All of this is not very efficient and we should directly expand $f(z)$ as a Laurent series in order to find the residue. Therefore,

$$f(z) = \frac{1}{z^5} \sum_{k=1}^{\infty} \frac{z^k}{k!}$$

and so we only need to find the term when $k = 4$. Therefore,

$$\text{Res}(f, 0) = \frac{1}{4!}$$

and

$$\oint_\gamma f(z)dz = \frac{i\pi}{12}.$$

So how do we decide between the two? You should assess if expansion is difficult.

Example. Let

$$f(z) = \frac{(\text{Log}(z))^2}{(z^2+4)^2} = \frac{(\text{Log}(z))^2}{(z-2i)^2(z+2i)}.$$

Thus, f is holomorphic on $\mathbb{C} \setminus (\{2i, -2i\} \cup \mathbb{R}^-)$. Thus, in order to compute the integral, we must find the residue of f at $2i$. Thus,

$$f(z) = \frac{\phi(z)}{(z-2i)^2}$$

with

$$\phi(z) = \frac{(\text{Log}(z))^2}{(z+2i)^2}$$

and $\phi(2i) \neq 0$. Therefore,

$$\text{Res}(f, 2i) = \phi'(2i) = \ldots$$