Notes on Electromagnetism

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Lectures by Professor David Hogg

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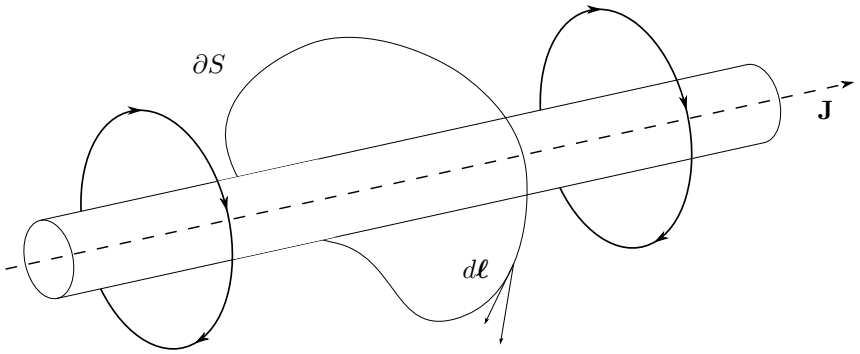

$$\oint_{\partial S} \mathbf{B} \cdot d\boldsymbol{\ell} = \iint_S \left(\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \tfrac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{A}$$

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Introduction

This is a set of lecture notes intended to provide a cursory glance at the essentials of Electromagnetism, based on my own arbitrary criteria. It is a work based in part on lectures given in Spring of 2020 by Professor David W. Hogg, the textbook *Electricity and Magnetism* by Purcell & Morin, the notes entitled "Electromagnetism" by David Tong, and the notes entitled "Part A Electromagnetism" by James Sparks. Special thanks is also in order to Cedric Yu, Randy Kayser, and Jinjie Zhang for being brilliant teachers.

A rudimentary command of vector calculus is assumed, mainly gradients, vector fields, and multivariable integration. A quick glance on Wikipedia would suffice, as well as leafing through any high-school textbook on the subject.

1 Electrostatics

1.1 Point Charges

It is an axiom of nature that elementary particles posess a quality called *electric charge* (denoted q or Q). This fundamental property determines how a given particle will interact electromagnetically; analogous to how a particle with mass interacts with other massive particles, particles with charge play the "game" of electromagnetism. Unlike mass, however, charges have a sign associated with them, which further determines how they interact.

In SI units, electric charge is measured in *Coulombs* (denoted C). An experimental fact is that protons and electrons carry charges $\pm e$ respectively, where $e = 1.6022 \times 10^{-19}$ C. All charges found in nature are integer multiples of e; for the purposes of these notes, however, we can say that $q \in \mathbb{R}$, while still remaining very presice. Note that charge can also be 0 (in the case of the neutron, for instance), which means that it is not affected by electromagnetic forces.

As you may have guessed, due to the small nature of these particles, we may reasonably approximate them as points, giving rise to the term *point charge*.

1.2 Forces and Coulomb's Law

Much like the fundamental interaction between mass is the familiar gravitational force, the interactions between charges at rest can be described by Coulomb's Law.

Given two charges q_1, q_2, and two position vectors $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^3$ respectively, the charge q_1 experiences an *electrical force* (denoted $\mathbf{F}_1$) due to the second charge:

$$\mathbf{F_1} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|^3} (\mathbf{r}_1 - \mathbf{r}_2) \tag{1.1}$$

The constant ϵ_0 in the denominator is called *vacuum permittivity*. In SI units,

$$\epsilon_0 = 8.854 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{Nm}^2}$$

Without loss of generality, we can put $\mathbf{r}_2$ at the origin, set $r = \|\mathbf{r_1}\|$, and set $\hat{\mathbf{r}} = \mathbf{r}_1/r$. We can therefore obtain the simpler equation

$$\mathbf{F_1} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} \tag{1.2}$$

Note that the force is proportional to the product of the charges, so that charges of opposite sign attract, while like charges repel; moreover, the force acts in the direction of the vector joining the two charges, and it is inversely proportional to the square of the distance between them.

It must be noted, before going further, that charged particles do indeed have mass, and therefore experience a gravitational force; however, it can be shown via Coulomb's law and Newton's law for Gravitation that the electromagnetic force is roughly 1×10^{42} times stronger than gravity! This means that to a good approximation, gravitational forces can be ignored[1].

One might begin to wonder: what occurs when there are multiple charges? Suppose we have three charges, q_1, q_2, and q_3; it is not immediately clear how they should interact. An important principle is that of *superposition*, which stipulates that the force with which two charges interact is not affected by the presence of a third charge. In essence, we are allowed to add the forces together and, to a very good approximation, it is true (there is a domain of very small distances for which superposition breaks down).

[1] On the interplanetary scale, gravitational forces win out because celestial objects have roughly equal amounts of positive and negative charge.

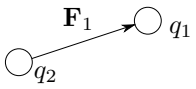

Figure 1: Repulsion between two particles.

We may therefore make the following generalization of coulomb's law. If we have charges $q_1, q_2, \ldots, q_n$ at positions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, then an additional charge q at position $\mathbf{r}$ experiences the force

$$\mathbf{F} = \sum_{i=1}^{n} \frac{qq_i}{4\pi\epsilon_0} \frac{(\mathbf{r} - \mathbf{r}_i)}{\|\mathbf{r} - \mathbf{r}_i\|^3} \tag{1.3}$$

In other words, the total force is the *vector sum* of all the others.

1.3 The Electric Field

What follows may seem like a trivial observation, but is actually a profound shift in perspective. We define the electric field at the point $\mathbf{r}$ of a single point charge q as follows:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{1.4}$$

Where r is the distance from q to the position vector $\mathbf{r} = (x, y, z)$, and $\hat{\mathbf{r}}$ points in the direction between q and the position vector. As you might expect, the field due to any discrete arrangement of charges is given by

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{n} \frac{q_i}{4\pi\epsilon_0} \frac{(\mathbf{r} - \mathbf{r}_i)}{\|\mathbf{r} - \mathbf{r}_i\|^3} \tag{1.5}$$

More practically, the force a point charge q experiences at position $\mathbf{r}$ is

$$\mathbf{F} = q\mathbf{E}(\mathbf{r}) \tag{1.6}$$

From which it is immediately apparent that the field $\mathbf{E}$ has units of Newtons/Coulombs.

Theories of modern physics are no longer especially concerned with forces, but rather the interplay between particles and fields. The advantage is that we can instead study the evolution of fields over time; by not having any instantaneous interactions, we can introduce forces that are compatible with the theory of special relativity. The electric field is thus a fundamental object in electromagnetism.

Regardless, we can draw *electric field lines* by asking what the behavior of a test charge q_0 might be when placed in the field. For instance, given a single charge of $+q$, we could imagine that another positive charge introduced will fly off in a straight line radially outward from the field source.

1.4 Continuous Charge Distributions

So far, we have only covered the discrete case of charges, where we approximate them by infinitessimal points. What if the charge is distributed accross a volume according to some density function? That is, there is a density $\rho = \rho(\mathbf{r})$ and a tiny volume dV such that $\rho dV \approx Q$, where Q is the total charge contained in dV. If the reader is at all familiar with calculus, it becomes apparent that one can find the total charge in a continuous distribution by taking the limit as $dV \to 0$ and the integral over the entire region:

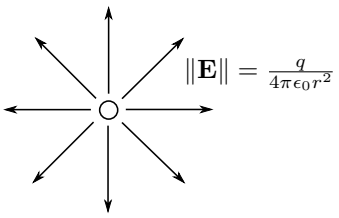

Figure 2: Electric field due to a spherical charge $+q$

$$Q = \iiint_V \rho(x, y, z) dV \tag{1.7}$$

where Q is the total charge contained in the volumetric region. It is conventional to use ρ to represent Coulombs/Volume, σ to represent Coulombs/Area, and λ to represent Coulombs/Length.

1.5 Electric Flux and Gauss' Law

The relationship between the electric field and its sources can be summarized by *Gauss' Law*. In order to conceptually understand it, we must first describe a quantity called *electric flux*.

In a word, electric flux is the rate of flow of electric field per area. Those familiar with multivariable calculus know that a normal vector can be associated with a plane of area dA; the corresponding normal vector is $d\mathbf{A}$.

Imagine, for the sake of argument, a small plane that is suspended in a fluid with velocity $\mathbf{v}$. We can model the rate of flow through this tiny patch of area as $\mathbf{v} \cdot d\mathbf{A} = \|\mathbf{v}\| \|d\mathbf{A}\| \cos\theta$, where theta is the angle between the velocity vector and the normal vector. In the case of electric fields, $\mathbf{v}$ is replaced by the vector $\mathbf{E}$, which gives the direction and magnitude of the electric field at any given point.

$$\Phi = \mathbf{v} \cdot d\mathbf{A} = \|\mathbf{v}\| \|d\mathbf{A}\| \cos\theta$$

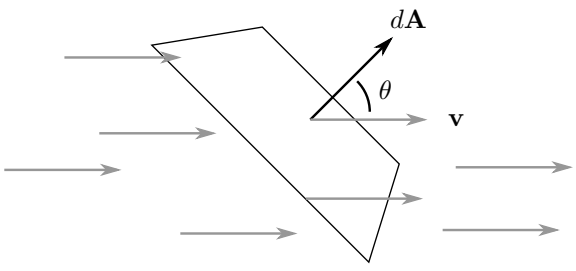

Figure 3: Flux through a small area $d\mathbf{A}$ with vectors $\mathbf{E}$.

Now imagine we have an arbitrary (piecewise differentiable) volume V in $\mathbb{R}^3$. Its boundary ∂V can be approximated by increasingly many little patches of area on the surface. In the limit, we can

find the flux across the entire surface by adding the flux accross all these infinitely small patches of area:

$$\Phi = \iint_{\partial V} \mathbf{E} \cdot d\mathbf{A} \tag{1.8}$$

where ∂V is the boundary of the surface.

Let us take a point charge Q, and consider a sphere of radius r centered around it. What is the flux Φ out of this surface?

The answer is simple when you recognize that $\mathbf{E}$ and the normal vector $d\mathbf{A}$ are always parallel to each other; that means the quantity $\mathbf{E} \cdot d\mathbf{A} = EdA$, where E is the magnitude of the field and dA is the area of the infinitessimal patch. We are left with

$$\Phi = \iint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = E \iint_{\partial V} dA.$$

Integrating the magnitude of the little patches over the surface just leaves us with the area of the sphere, which we all know from geometry is $4\pi r^2$. We are also familiar with the field generated by a single point charge q. Therefore,

$$\Phi = E(4\pi r^2) = \frac{q}{4\pi\epsilon_0 r^2}(4\pi r^2) = \frac{q}{\epsilon_0}.$$

Amazinlgy, the flux throughout the entire surface does not depend on the radius of the sphere we use! This makes sense, because even though the area of a larger sphere increases proportionally to r^2, the electric field gets weaker with radius by the same amount, and so these effects cancel out. With this in mind, it is not so farfetched to claim that the flux does not depend on any surface that we use. In other words, flux is *invariant with respect to surface choice*. This can be proven mathematically by the divergence theorem, which will not be explained here.

Regardless, this gives rise to a powerful new law, known as *Gauss' Law*. It states that the flux through any closed surface is equal to the charge enclosed by that surface over the constant of vacuum permittivity. Mathematically:

$$\Phi = \iint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0} \tag{1.9}$$

where ∂V is the surface chosen (known as the Gaussian Surface), and Q is the total charge enclosed in the volume V. Note that if there is no charge enclosed, the charge will enter and exit the boundary at the same rate, meaning that there is no net flux across the boundary.

This integral relationship between Gauss' law also suggests a differential relationship, whereby we in a sense "take the derivative" of both sides. To see what we mean by this, Gauss' law is equivalent to

$$\iint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = \iiint_V \nabla \cdot \mathbf{E} dV \tag{1.10}$$

where $\nabla \cdot \mathbf{E}$ is a scalar quantity known as the *divergence* of a vector field, and the equality is given by the *divergence theorem*. This measures how much a vector function spreads out, or, more presicely, the volume density of net outward flux; it is defined as finding the flux over increasingly shrinking volumes. From this equality, we see that

$$\iiint_V \nabla \cdot \mathbf{E} dV = \frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0} \iiint_V \rho dV$$

where ρ is the volumetric charge density, and taking the integral over V gives us the total charge Q. We can "cancel out" the integral on both sides, giving us the *differential form of Gauss' Law*:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}. \tag{1.11}$$

The power of equation (1.9) is that it allowa us to calculate electric fields of objects that are shaped in a variety of ways. To do this, we simply pick an arbitrary volume such that the field of the object is always parallel (or normal) to the object's boundary; we can then remove the E term from the integral, use familiar geometric formulas to find the area of the surface, and solve an equation for the magnitude of the field $E(r)$. We will now consider some examples.

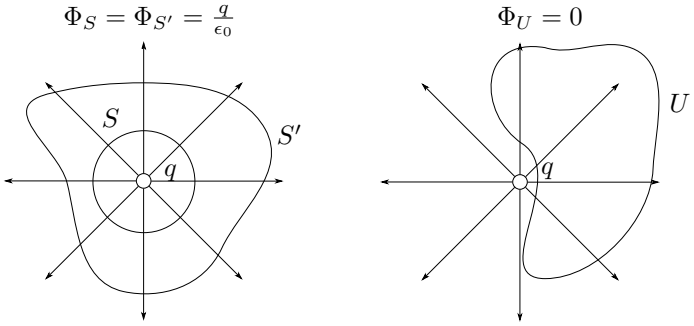

Figure 4: Flux through S equals flux through S'. There is no charge enclosed in U, so the total flux is 0.

1.5.1 Field of Spherical Charge

Suppose we have a solid sphere V of charge with radius r, with a uniform density ρ that only depends on radial distance. Due to the high symmetry of the problem, the choice of Gaussian surface is obvious; we must chose another sphere with a larger radius, say R.

Let Q equal the total charge of the sphere, found by using equation (1.7). Using the formula, we get that

$$E(R) = \frac{1}{4\pi\epsilon_0 R^2} \iiint_V \rho dV = \frac{Q}{4\pi\epsilon_0 R^2}.$$

Therefore, at a distance R from the center, we can think of all the charge Q as if it were a point charge located at the center.

Further still, what does the field look like inside the sphere? Consider again a sphere of radius R, and Gaussian surface of radius r. For $r \geq R$, it is like all the charge is located at the center of the sphere, and the total charge is the volume times the density.

$$\iint_S \mathbf{E} \cdot d\mathbf{A} = \frac{\rho(4\pi R^3/3)}{\epsilon_0}$$

$$E(r) = \frac{\rho(4\pi R^3/3)}{4\pi\epsilon_0 r^2} = \frac{\rho R^3}{3\epsilon_0 r^2} \tag{1.12}$$

If $r \leq R$, the charge on the outside can be ignored, as it is not enclosed in our Gaussian surface of radius r. Considering only the charge on the inside, we see that

$$\iint_S \mathbf{E} \cdot d\mathbf{A} = \frac{\rho(4\pi r^3/3)}{\epsilon_0}$$

$$E(r) = \frac{\rho(4\pi r^3/3)}{4\pi\epsilon_0 r^2} = \frac{\rho r}{3\epsilon_0} \tag{1.13}$$

The field increases linearly inside the sphere, because the cubic gains from the volume outweigh the quadratic weakening of the field. Outside of the sphere, we resume the familiar $\sim 1/r^2$ law.

1.5.2 Field of Linear Charge

Consider a very thin and very long (i.e., infinite) charged wire with uniform charge density λ. Due to the symmetry of the problem, each field line is directed radially away from the wire. Heuristically, because the wire is infinite, for each charge that creates a field vector pointing to the right, there is a charge that produces the same vector pointing to the left. The horizontal components of the field cancel, and we find that we are left with only the vertical direction. A reasonable choice of closed surface could be a cylinder of radius r and length ℓ that wraps around the wire.

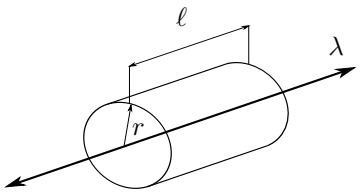

Figure 5: On the tube, $\mathbf{E}$ is parallel to $d\mathbf{A}$.

The surface area of a cylinder is $2\pi r^2 + 2\pi r\ell$. At the "caps" of the cylinder (i.e. the top and bottom circles), we observe that $d\mathbf{A}$ is pointing parallel to the wire, but perpendicularly to the field. Because $\mathbf{E} \cdot d\mathbf{A} = 0$ at the caps, they don't contribute to the integral. We only need to consider the area of the cylinder that wraps around, which has area $2\pi r\ell$.

$$(E)(2\pi r\ell) = \frac{\lambda\ell}{\epsilon_0}$$

$$E(r) = \frac{\lambda}{2\pi\epsilon_0 r}. \tag{1.14}$$

1.5.3 Field of Planar Charge

Consider an infinite plane of charge, with uniform charge density σ. Surely, the electric field points perpendicularly to the plane; there is no other unique direction in the system. Gauss tells us that we can draw a cylinder that pierces the plane, with the only area vectors parallel to the cylinder at the caps. Thus, we only need to take the top and bottom areas into account. Note that even though the field on one side points in the direction opposite of the other, so does the corresponding area vector. We're effectively subtracting a negative.

$$E(\pi r^2) - (-E(\pi r^2)) = \frac{\sigma\pi r^2}{\epsilon_0}$$

The charge enclosed is the cross-sectional area of the cylinder, which is a circle of radius r times the charge density σ.

$$E(r) = \frac{\sigma}{2\epsilon_0} \tag{1.15}$$

$$A = \pi r^2$$

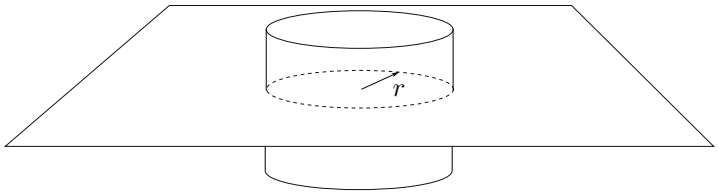

Figure 6: On the plane, **E** points up on top and down on the bottom, if the charge density is positive.

2 Electric Potential

2.1 A Brief Review of Line Integrals

Suppose you have a vector field $\mathbf{F}$, and you trace out a curve $C = \mathbf{c}(t)$ in space. How much of the field $\mathbf{F}$ do we "accumulate" on this path? Physically speaking, its as if we were in a force field, and we wanted to know how much energy we have to expend moving a mass (or charge) along a curve.

The key intuition is dividing the path into infinitessimally small vectors $d\mathbf{s}$, each locally tangent to C, and summing up the component of $\mathbf{F}$ in that direction. Informally speaking, we measure how much $\mathbf{F}$ and the curve C are parallel to one another.

In order to achieve this, we parametrize a curve $\mathbf{c}(t)$ such that the endpoints $\mathbf{c}(a)$ and $\mathbf{c}(b)$ give the endpoints of the curve C.

Clearly, $\mathbf{c}'(t)$ gives a vector that is locally tangent to C; Note that as we consider finer and finer differences of t, we more closely approximate the curve with tangent vectors. In the limit, we obtain the expression

$$S = \int_C ds = \int_a^b \|\mathbf{c}'(t)\| dt$$

Where S is the total length of C. With this in mind, we want to find an expression that finds the component of $\mathbf{F}$ along the entire length of C (If C and $\mathbf{F}$ are antiparallel, we expect a negative contribution to the integral. In the case of arc length, we don't care about the direction of $d\mathbf{s}$, only the magnitude).

We call the following integral a *line integral*:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt. \tag{2.1}$$

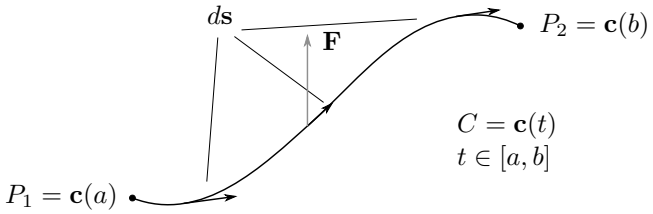

Figure 7: We measure the component of $\mathbf{F}$ along $d\mathbf{s} = \mathbf{c}'(t)dt$ over the entire curve.

What if the vector field $\mathbf{F}$ is the gradient of some scalar function, say g? That is, $\nabla g = \mathbf{F}$. Remember that the output of the gradient operator on a function is a vector field.

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \nabla g(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

According to the chain rule for vector-valued functions,

$$\nabla g(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \frac{dg(\mathbf{c}(t))}{dt}$$

The expression in the integral then becomes

$$\int_a^b \frac{dg(\mathbf{c}(t))}{dt} \cdot dt = \int_a^b dg(\mathbf{c}(t)) = g(\mathbf{c}(b)) - g(\mathbf{c}(b)) \tag{2.2}$$

by the fundamental theorem of calculus.

The implications of this result is huge. It states that if the vector field is the gradient of some scalar function, then the path we take doesn't matter; only the endpoints do! This allows us to define a quantity between two points, without needing to specify a path. With that said, we can draw an equivalent conclusion: the integral of a gradient vector field around a closed loop is 0. Take any two different paths, C_1 and C_2 from point P_1 and point P_2. Clearly, they both point in the same direction (from the starting point to the ending point). from (2.2), we have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s} \tag{2.3}$$

Which is equivalent to

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_2} \mathbf{F} \cdot d\mathbf{s} = 0 \tag{2.4}$$

So we take two paths, which start and end at the same points, reverse the direction of C_2 so it travels from point P_2 to point P_1, and the integral of that will be zero. Physicists sometimes put a circle around the integral, in order to indicate it is done around a closed loop:

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = 0 \tag{2.5}$$

However, not all electric fields obey the property that they are gradients of a scalar function, as we will later see. Fields with static charges, however, do obey this property.

2.2 The Potential Function

In the last section, we mentioned a quantity that can be defined by specifying two points in an electrostatic field. We call this function the *electric potential difference* between two points:

$$\Delta\phi = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{s}. \tag{2.6}$$

This is the work per unit charge that an external force does moving a positive charge from P_1 to P_2. The external force must supply a $\mathbf{F}_{\text{external}} = -q\mathbf{E}$ to balance the electrical force $\mathbf{F}_{\text{electric}} = q\mathbf{E}$, so we put a minus sign.

If we like, we can fix P_1 and consider how the potential varies as P_2 moves around, so we can simply write the potential as a function $\phi(\mathbf{r})$ where $\mathbf{r}$ is the position vector—as long as we remember which reference point we agreed upon. By convention, the point P_1 of zero potential is at infinity, which makes sense; the farther out you go, the weaker the electric field becomes. The value of ϕ is a scalar, and it is a function of position.

Working out the units of this quantity, we get Energy/Charge, or Joules/Coulombs. The unit has a name of its own, called the *volt*:

$$1 \text{ Volt} = \frac{1 \text{ Joule}}{1 \text{ Coulomb}}. \tag{2.7}$$

It takes 1 Joule of energy to move a coulomb of charge through a potential difference of 1 volt. Do not confuse potential difference with potential energy; the *potential energy* of a charge distribution is the force required to bring all the charges together, with all of them starting infinitely far apart.

For instance, if we fixed the origin at a charge q_1, what is energy it takes to bring another charge in at a radius R from q_1?

The potential energy can be thought of as the work done to bring q_2 in from infinity:

$$U(R) = \int_R^\infty \mathbf{F} \cdot dr = -q_2 \int_\infty^R \mathbf{E} \cdot dr$$

Note that we swapped the bounds of integration, so the expression assumes a minus sign. We know the field due to a point charge q_1, so we obtain

$$U(R) = q_2 \int_\infty^R \frac{q_1}{4\pi\epsilon_0 r^2} dr$$

$$U(R) = q_2 \left(\frac{q_1}{4\pi\epsilon_0 R} \right) = q_2 \phi \tag{2.8}$$

Where ϕ is the potential difference between R and infinity.

If we would like to assemble multiple charges together, it is clear that the order in which we do it does not matter, because order doesn't matter in the case of two charges, and the forces are additive, it shouldn't matter for N charges.

If we have charges $\{q_1, \ldots, q_N\}$, the work required to assemble them is

$$U = \frac{1}{2} \sum_{j=1}^{N} q_j \sum_{k \neq j} \frac{1}{4\pi\epsilon_0} \frac{q_k}{r_{jk}} \tag{2.9}$$

Where r_{jk} is the distance between charge q_j and charge q_k. There is a corrective factor of $1/2$ because we count each interaction $U_{AB} = U_{BA}$ twice for charges A and B.

In general, the work required to assemble a continuous charge distribution is

$$U = \frac{1}{2} \iiint_V \rho(\mathbf{r}) \phi(\mathbf{r}) dV \tag{2.10}$$

2.3 Electric Field as Gradient Field

We can observe from equation (2.6) that the electric field is some kind of derivative of the potential function; more presicely, the electric field is the *gradient* of the potential function. If you remember from a course on vector calculus, the gradient ∇f of a multivariable function f is a vector which tells us how f varies in the neighborhood of a point. It is defined as

$$\nabla f := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \tag{2.11}$$

Because of how it's defined, it always points in the direction of steepest ascent. It is the direction one must move to increase f the fastest. Say you're on a mountain (modeled by a height function $h(x, y)$); ahead of you, the mountain slopes up, and behind you it slopes down. There is always a direction in which a short step will take you higher than any other step of the same length in another direction. If you're at the peak of the mountain, any step you take will lower you, so the gradient will be the zero vector.

In order to show the other differential relationship between the potential and the electric field, consider the value of $\phi(\mathbf{c}(t))$ at two nearby points, (x, y, z) and $(x+dx, y+dy, z+dz)$. By the chain rule, the infinitessimal change in ϕ due to an infinitessmial change in t is measured as

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \tag{2.12}$$

Under the assumption that the partial derivatives are continuous. Notice how this can be reformulated into a dot product:

$$d\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \cdot (dx, dy, dz) \tag{2.13}$$

However, we know that from equation (2.3),

$$d\phi = -\mathbf{E} \cdot d\mathbf{s} \tag{2.14}$$

Notice how the right side of the dot product in (2.13) is simply $d\mathbf{s}$. The left side of the dot product is simply $\nabla\phi$ from (2.6). Therefore, we arrive at the important concluson that

$$\mathbf{E} = -\nabla\phi. \tag{2.15}$$

Instead of pointing in the direction of greatest ascent, the minus sign ensures that $\mathbf{E}$ points in the direction that will *decrease* the potential the fastest.

2.4 Equipotential Curves

Just like on mountains one can find paths around it which have constant elevation, we can find curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$ that have constant potential. Such surfaces are called *equipotential surfaces*, and can be used in drawing diagrams.

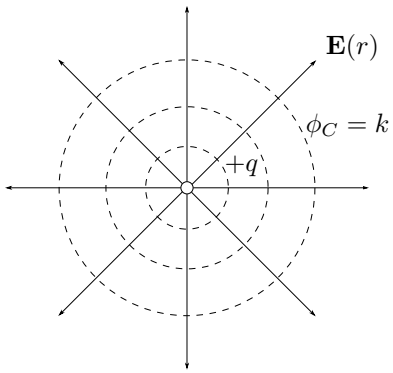

Figure 8: For a charge $+q$, the surface of every sphere cented around $+q$ is equipotential. The field lines are always perpendicular to each surface.

What is the geometric relationship between equipotentials and field lines? In the above figure, it is apparent that the electric field lines are normal to each equipotential; however, this is always the case, no matter what the distribution of charge is! This is because the gradient measures the direction of steepest ascent. If the gradient were not perpendicular to the surface, that means that it would have some component on the surface. This means the potential would change in that direction on the surface—but then it wouldn't be an equipotential! We can construct an informal proof of

why this is, using the chain rule for parametrized functions. Suppose ϕ is constant along the curve $C = \mathbf{c}(t)$. That is, $\phi(\mathbf{c}(t)) = k$. Taking the gradient,

$$\nabla\phi(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = 0.$$

The vector $\mathbf{c}'(t)$ is really the vector tangent to C at value t. As we know, in $\mathbb{R}^3$, if the dot product between two vectors are zero, then they are perpendicular to one another.

If the charge is located in a finite volume, then the farther out one travels, the more the equipotential surfaces resemble spheres. This is because from really far away, finite volumes resemble points.

2.5 Potential of a Continuous Charge Distribution

Recall from equation (2.8) the potential associated with a single point charge q is just

$$\phi = -\int_r^\infty \frac{q}{4\pi\epsilon_0(r')^2} dr' = \frac{q}{4\pi\epsilon_0 r} \tag{2.16}$$

where r is the radial distance from q and we have elected the points of zero potential to be infinitely far away. The principle of superposition, elaborated upon in section 1.2, tells us that fields, and by extension potentials, are additive in nature. We are allowed to sum the potentials provided they all agree on a reference point; if the charges are enclosed in some finite volume V, then we can consistently elect the point of zero potential to be at an infinite radius.

Under these conditions, the potential of any continuous charge distribution that can be enclosed in a finite volume is

$$\phi(x, y, z) = \iiint_V \frac{\rho(x', y', z')}{4\pi\epsilon_0 r} dV \tag{2.17}$$

where r is the distance from the volume element dV to the point (x, y, z) at which the potential is being evaluated. That is, $r = \|(x - x', y - y', z - z')\|$. From the scalar potential function we can always find the electric field by taking the negative gradient. In that sense, it is easier to compute the potential everywhere and then compute the field, rather than compute the field everywhere and integrate to find the potential.

2.5.1 Potential of a Linear Charge

Because this only works for finite distributions of charge, we can see the difficulty that arises in evaluating the potential for, say, an infinite wire. Trying to take the integral in equation (2.12) results in a divergent integral (feel free to test this yourself, by integrating from $-\infty$ to ∞, and substituting $\rho(x', y', z')$ with $\lambda(x)$).

Therefore, we must choose a reference point that is a finite distance away, at point P_1 at a distance of r_1 from the wire. Using the equation (2.3) for potential,

$$\phi_{21} = -\int \mathbf{E} \cdot d\mathbf{s} = -\int_{r_1}^{r_2} \left(\frac{\lambda}{2\pi\epsilon_0 r}\right) dr = -\frac{\lambda}{2\pi\epsilon_0} \ln r_2 + \frac{\lambda}{2\pi\epsilon_0} \ln r_1$$

We can re-write this equation as

$$\phi = -\frac{\lambda}{2\pi\epsilon_0} \ln(r) + C \tag{2.18}$$

We can make the expression on the right a constant, because the chosen reference point does not vary, only r_2 does. The extra constant term does not change the negative gradient of this function with respect to r, and we get the familiar equation for a field of a linear charge distribution:

$$E = -\nabla\phi = -\hat{\mathbf{r}}\frac{d\phi}{dr} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}}.$$

2.6 Dipoles

A useful setup to consider is that of a *dipole*, with two equal and opposite charges $\pm q$ located at positions $\pm a$ on the y-axis, as shown in Figure 8.

Although we could write an exact expression for the potential, it is much more interesting to consider the approximation under large distances away from the dipole.

We must first note that the dipole is rotationally symmetric about the y-axis, so it suffices to find the potential in an arbitrary plane containing the axis. The planar case allows us to express the point P in terms of the ordered pair (r, θ), where θ is measured down from the vertical axis.

The potetial at the point P is

$$\phi_P = \frac{q}{4\pi\epsilon_0 r_1} - \frac{q}{4\pi\epsilon_0 r_2}$$

Should one require, the exact potential can be calculated using the law of cosines. In the limit $r \gg 2a$, however, observe that the two lines from the charges are essentially parallel, so $r_1 = r - a\cos\theta$ and $r_2 = r + a\cos\theta$.

$$\phi = \frac{q}{4\pi\epsilon_0(r - a\cos\theta)} - \frac{q}{4\pi\epsilon_0(r + a\cos\theta)}$$

Using the approximation $1/(1 \pm \varepsilon) \approx 1 \mp \varepsilon$, we see that

$$\begin{aligned}\phi = \frac{q}{4\pi\epsilon_0 r}\left(\frac{1}{1 - \frac{a\cos\theta}{r}} - \frac{1}{1 - \frac{a\cos\theta}{r}}\right) &\approx \frac{q}{4\pi\epsilon_0 r}\left[\left(1 + \frac{a\cos\theta}{r}\right) - \left(1 - \frac{a\cos\theta}{r}\right)\right] \\ &= \frac{2qa\cos\theta}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \qquad (2.19)\end{aligned}$$

Where p is called the *dipole moment* and is defined to be the product of $2a$ and q.

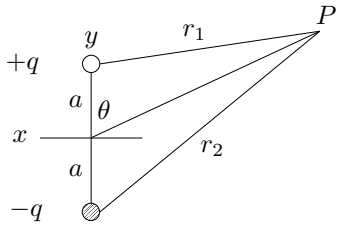

Figure 9: Finding the potential ϕ at point P, under the condition $r \gg 2a$.

It is important to note that the potential for a dipole falls off like $\sim 1/r^2$, instead of the potential for a point charge which is $\sim 1/r$. This makes sense, because the potentials for two opposite point charges nearly cancel; in the limit as $a \to 0$, the potential everywhere becomes 0, because the charges cancel out.

Now that we have the potential, we are able to compute $\mathbf{E}$ everywhere.

$$\mathbf{E}(r,\theta) = -\nabla\phi = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{2qa\cos\theta}{4\pi\epsilon_0 r^2}\right)\hat{\mathbf{r}} - \frac{\partial}{\partial\theta}\left(\frac{2qa\cos\theta}{4\pi\epsilon_0 r^2}\right)\hat{\boldsymbol{\theta}}$$

Note that taking the gradient in shperical cordinates requires the extra term $1/r$. Therefore, the electric field everywhere is

$$\mathbf{E}(r,\theta) = \frac{2qa\cos\theta}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) = \frac{p\cos\theta}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \tag{2.20}$$

It is easy to see that the field points upward everywhere on the y-axis, by taking $\theta = 0, \pi$. On the x-axis, the field points in the positive tangential direction for $\theta = \pi/2$, and in the negative tangential direction for $\theta = 3\pi/2$.

3 Conductors and Capacitors

So far, we have learned about fields and potentials in cases where charges are fixed in their positions. We will now turn to the cases of charges that are free to move around.

Broadly speaking, materials can be separated into two main categories: *insulators* and *conductors*. These classifications are a function of how mobile electrons are within the material. This is why electrical wires are made out of metal and not rubber; metals are better able to transport electrons from one place to another.

3.1 Conductors in Electrostatic Fields

This section is devoted to analyzing conductors in electrostatic fields. Normally, when subject to a field, the charges in the conductor re-distribute in order to counteract the forces to which they are subject. This normally occurs within a fraction of a second. In a static situation, one concludes that the field inside of it must be 0. If that were not the case, then surely the charges would move around, and we would no longer be dealing with a static situation (note that this only holds true in the ideal case of a homogeneous, isotropic conducting material. It is still a profoundly applicable conclusion to draw). Of course, the electric field is not zero when we zoom in on a proton; but on average, the electric field in a conducting material is 0.

The charges are also limited in the sense that they cannot leave the material; this ensures that the final distribution of charge at the surface of the conductor is such that its field and the external field presicely cancel. The total charge of the conductor, denoted Q, is fixed because there is no way for charge to "fall off."

Having concluded that $\mathbf{E} = 0$, it stands to reason from the differential form of Gauss' law, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, that the volumetric charge density also vanishes in the interior. Of course, as with the field, this holds only in an average sense.

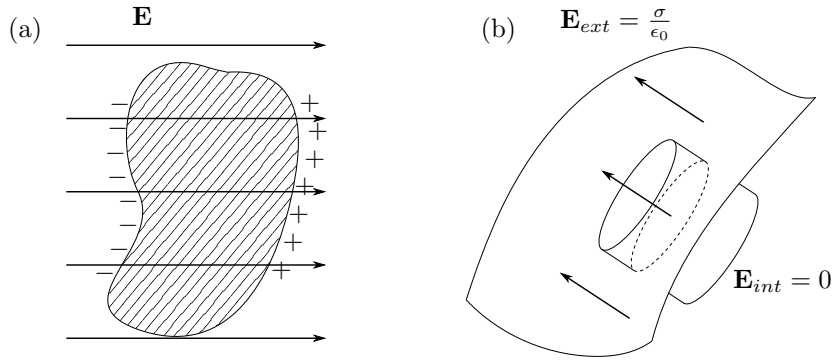

Figure 10: (a): Charges moving to create a net $\mathbf{E} = 0$ inside the conductor. (b): A Gaussian surface demonstrating the electric field strength at the surface of a conductor.

With our previous knowledge, we are allowed to make several implicit statements about perfect conductors:

1. Inside of the conductor (i.e., the metal inside), we must have $\mathbf{E} = 0$. If not, then charges would be subject to forces and move around.
2. Equivalently, we determine that $\rho = 0$ inside of a conductor, via the differential form of Gauss' law.

3. Since $\mathbf{E} = 0$ in a conductor, and $\phi = \int \mathbf{E} \cdot d\mathbf{s}$, We find that the electrostatic potential ϕ must be constant inside and on the surface of a conductor.

4. Conductors can be electrically neutral, meaning that they have positive and negative charges which balance out. Alternatively, they may have a net charge; in which case, all the charge is distributed across the surface of a conductor.

5. Since ϕ is constant, the surface of any conductor is equipotential. Therefore, the electric field must be perpendicular to the surface. This also comes from the fact that if the field were not normal to the surface, then the surface charges would move.

6. Using Gauss' law, we can construct a Gaussian surface and find that the field just outside a conductor is $E = \sigma/\epsilon_0$, where σ is the local surface charge density.

3.1.1 Cavity in a Conductor

Suppose intsead of a solid mass of conductor, we have a conductor with a cavity in the inside. The conductor is, of course, subject to an external electric field of some strength. We know there is no field in the metal, but what about inside the cavity?

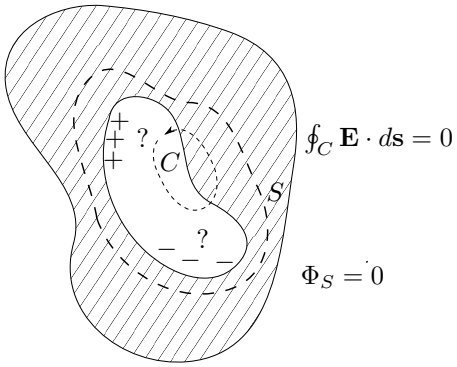

Figure 11: Technically, Gauss' law does not preclude the possibility of having equal amounts of positive and negative charge in S. However, we can use a closed loop C to show why this cannot be the case.

We will demonstrate that if the cavity is empty, there are no fields inside it, no matter the shape of the conductor or the cavity proper.

First, consider a surface S that encloses the cavity but stays inside the material everywhere. Because the field in the material is 0, the net flux across S is 0, so we know the total charge enclosed is zero. That is not to say, however, that there are no regions of net charge on the inner surface of the conductor; we cannot rule that out by Gauss' law.

However, we also can rule out the possibility of these regions, because such clusters of charge will repel themselves and eventually balance out. We can use equation (2.5) to demonstrate this fact; suppose there are positive and negative regions of charge; we can create a curve C that originates from the positive charges and terminates on the negative ones, and returns to the starting point by travelling through the conductor. Because the field in the material is 0, but the field from the

positive to the negative charges isn't zero, then the integral around the closed loop C is not zero, which is impossible. Therefore, there are no fields inside the empty cavity, and no net charges on the inner surface of the cavity.

This is a phenomenon known as *shielding*, because no static distribution of charge can ever create a field inside the cavity.

3.1.2 Point Charge in Cavity

Now suppose we have the same setup as in the last section, only we place a point charge q somewhere within the cavity.

Analogous to the previous example, we draw a surface S that encloses the cavity; we know from Gauss' law that this must contain zero charge, because there is no field inside the material of the conductor. Therefore, the total charge on the inner surface of the conductor must be $-q$.

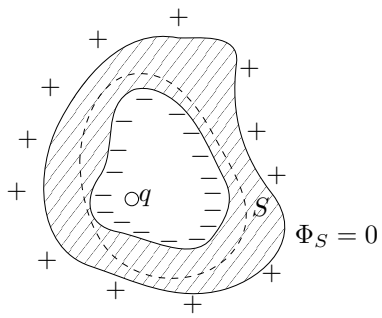

Figure 12: Since a total charge of $-q$ accumulates on the inner surface, the field on the outside is independent from the point charge's position.

If we create a Gaussian surface surrounding the entire conductor, we know that the total charge is q; the question is, how is this charge q distributed on the outer surface?

Suppose we only have the point charge and the negatively charged inner surface; clearly, there is zero field outside of them. Moreover, when we add back the rest of the metal (which has charge $+q$), we do not disturb the configuration of charges we already had. Therefore, the total charge on the outer surface of the conductor is *independent* of where the charge q is located in the interior. The field would be as if we removed the charge from the center and dumped a total charge q over the conductor (which will distribute itself on the surface in a particular way, according to the object's geometry).

As a matter of fact, they have to be distributed such that the potential on the surface is constant. Otherwise, there would be a potential difference across the conductor, meaning that there are fields inside it, which cannot occur.

3.2 General Electrostatic Equations

Recall the differential form of Gauss' law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

And the relationship between the electric potential and the electric field

$$\mathbf{E} = -\nabla\phi$$

We can combine these two equations into one form to obtain the differential equation the potential must obey

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \tag{3.1}$$

Where the operator ∇^2 is the *Laplacian*, and the above equation is known as *Poisson's Equation*. Written out in cartesian coordinates, it reads

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -\frac{\rho}{\epsilon_0}. \tag{3.2}$$

In general, the Poisson equation always has a solution of the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\mathbf{r'})}{\|\mathbf{r} - \mathbf{r}'\|}dV \tag{3.3}$$

where $\mathbf{r'}$ is the position of a volume element dV, and we divide the charge density at $\mathbf{r'}$ by its distance.

From a mathematical point of view, the classical theory of fields is simply a study of the solutions to this equation. We can therefore state the problem in terms of the potential ϕ; everywhere outside of the conductor, the potential has to satisfy $\nabla^2\phi = 0$, because there is no surface charge density enclosed. In cartesian coordinates, this is called *Laplace's Equation*, and a function that satisfies this equation is known as *harmonic*:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{3.4}$$

Of course, once we solve for ϕ, we can obtain the field by taking the gradient. The problem is to find a function that satisfies the Laplace equation and meets the specified conditions concerning conducting surfaces—conditions which might have been set in many different ways. Suppose we have multiple conductors, each of which must be held at potential ϕ_k. The region in which ϕ is defined is bound in its totality by these surfaces if we also include a surface "at infinity" whose potential must approach 0. Or, we could consider the enclosed region inside a conductor; either way, we are considering a "boundary-value problem," in which the value a function must take on a boundary is specified for the entire boundary (i.e., the potentials at the surface of various conductors).

It can be proven mathematically that for any system of the conductors, there is no more than one solution for this problem. From a physical point of view, it can be argued that ther must always exist one solution, because the system exists in that particular state, which must be the solution. All solutions are therefore unique.

Theorem 3.1 (Uniqueness of Solutions)**.** Assuming there is a solution for $\phi(x, y, z)$ with a given set of boundary conditions, then such a solution must be unique.

A corollary of this is:

Corollary. If the space inside a hollow conductor is empty of charge, then the electric field is zero within it.

This is because we know that the boundary of the problem specifies that the potential must be some constant ϕ_0, and the potential $\phi(\mathbf{r})$ must satisfy Laplace's equation, which stipulates that the Laplacian must be 0. Clearly, $\phi(\mathbf{r}) = \phi_0$ is a solution, so it must be *the* solution. This in turn implies that the field inside must be zero.

3.3 The Method of Images

Sometimes, calculating the field of an arbitrary configuration can be tedious at best and nearly impossible at worst. Fortunately, there are certain tricks that make such problems tractable, and even trivial in some cases. One of the most enduringly successful tricks is called "The Method of Images," whereby we use the symmetries of another problem we have solved to fit the boundary conditions of a particular one.

For instance, consider the case of a positive charge q separated by a perpendicular distance a from an infinite conducting plane which we assign with zero potential. What sort of field and charge distribution can be expected? We expect q to attract the negative charge on the plate, but we don't expect them to pile up at the foot of the perpendicular from q, because then there'd be a potential difference within the conducting metal, and that has been ruled out.

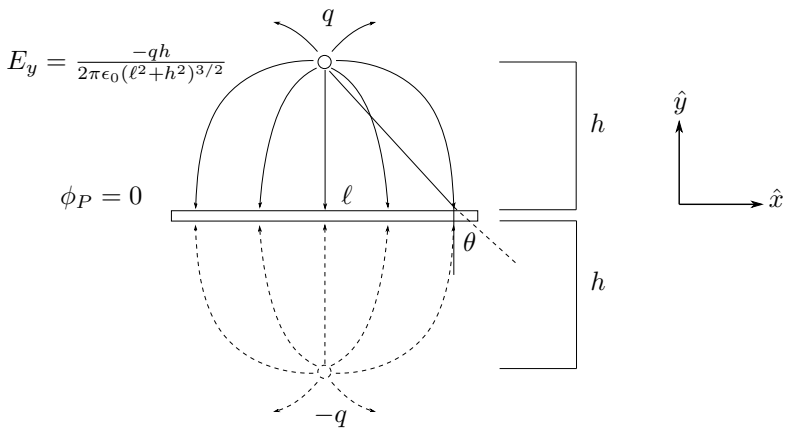

Figure 13: The boundary conditions of a point charge and an infinite plane are the same as those of the dipole. Thus, by the uniqueness theorem, the two systes must have equivalent solutions.

By the uniqueness theorem, we know that if we can find a problem similar enough such that the boundary-value constraints are met, then that must be *the* solution. One such problem we encountered is the electric dipole. We know our boundary constraints are that the conducting plane must be held at zero potential throughout; the dipole has an infinite plane of equipotentiality bisecting the two point charges. Moreover, there must be a charge $+q$ held at height h above the conducting plane. Therefore, the solution to the dipole equation must be the solution to this specific problem. The imaginary charge $-q$ is called the "image charge."

The field in the $\hat{y}$ direction for a dipole is given by

$$E_y = \frac{-2q}{4\pi\epsilon_0(\ell^2+h^2)}\cos\theta = \frac{-2q}{4\pi\epsilon_0(\ell^2+h^2)} \cdot \frac{h}{(\ell^2+h^2)^{1/2}} = \frac{-qh}{2\pi\epsilon_0(\ell^2+h^2)^{3/2}}.$$

Going back to the original setup, we are now able to calculate the surface charge density of the plane, which we know for a conductor is $E = \sigma/\epsilon_0$ (there is no field underneath the plane):

$$\sigma = \epsilon_0 E_y = \frac{-qh}{2\pi(\ell^2+h^2)^{3/2}}$$

integrating radially over the plane's entire surface (i.e., ℓ from 0 to ∞):

$$\int_0^\infty \sigma(2\pi\ell)d\ell = -qh\int_0^\infty \frac{\ell}{(\ell^2+h^2)^{3/2}} = \left[\frac{qh}{(\ell^2+h^2)^{1/2}}\right]_0^\infty = -q$$

which is what we expect; all the flux leaving the $+q$ charge ends up distributed along the plane in some manner (σ).

3.4 Capacitors

A large conducting plane has charge Q and a certain potential ϕ_0, with the convention of zero potential at infinity. The charge is proportional to the potential:

$$Q = C\phi_0 \tag{3.5}$$

We know this from the superposition principle; if we know the solution for one set of charges, and add the same configuration, the field strength is doubled, as is the potential. Therefore, potential is proportional to charge. We call this constant of proportionality C for capacitance. Charge is measured in coulombs, and potential is measured in volts. This gives rise to a new unit, called the *farad*:

$$1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt}} \tag{3.6}$$

The farad happens to be an enormous unit (this is because the coulomb is also quite large); the capacitance of an isolated sphere the size of the earth is only

$$C = \frac{Q}{\phi_0} = \frac{Q}{\frac{Q}{4\pi\epsilon_0 r}} = 4\pi\epsilon_0 r = 4\pi\epsilon_0 \cdot 6.4\times10^6 \text{ m} \approx 7\times10^{-4} \text{ farad!}$$

Consider two large metal plates of area A, parallel to one another and separated by a distance h, both with equal and opposite charge $\pm q$. This configuration is known as a "parallel-plate capacitor." The charges on one plate will be attacted by the ones on the other, and will therefore spread out uniformly on the inner surface of the plate. We can construct a Gaussian surface to find that the field in between is given by σ/ϵ_0.

This means that there is a potential difference between the two plates, and that is given by

$$V = \phi_{top} - \phi_{bottom}$$

This difference is the energy per unit charge needed to bring a test charge from the bottom plate to the top:

$$V = E \cdot h = \frac{\sigma}{\epsilon_0}h = \frac{qh}{A\epsilon_0} \tag{3.7}$$

And because $Q = C\phi_0$, we can obtain an equation for the capacitance:

$$C = \frac{\epsilon_0 A}{h} \tag{3.8}$$

Of course, this is mereley an approximation; the closer one goes to the edges of the planes, the more bent the field lines are, and so we must correct for the field.

Notice how charge does not factor into this equation at all; it is simply a function of the geometry of the conductor, the distance between them, and the so-called "dielectric," or the material through which a field travels. Capacitance therefore measures the ability of a conductor to store charge. When we talk of the charge q stored in a capacitor, we really assume that they have equal and

opposite charges. This is not unreasonable; if you hook up a battery between the two conductors which initially don't have any charge, the charge that leaves one will end up on the other.

Technically speaking, any arrangement of two conductors can be considered a capacitor, but it just so happens that the most ubiquitous are of the parallel-plate variety. The distance between them can be made arbitrarily small and the area of each arbitrarily large; this in turn will yield a system which can store lots of charge without making too large a potential difference[2].

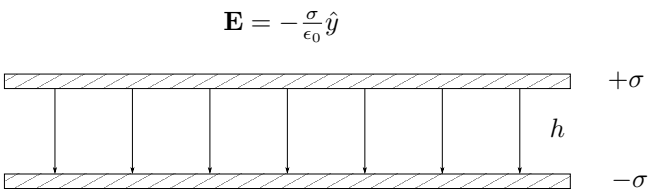

Figure 14: Far away from the edges, the electric field inside a parallel plate capacitor is essentially uniform.

Interestingly enough, one application of these capacitors is to make *positrons*, which are elementary particles that have the same mass as electrons but opposite charge; in other words, it is the antimatter counterpart of the electron. If we have a capacitor with sufficiently large charge on either end, a positron and electron will be created out of the vacuum, and jump to the plate of opposite charge, thereby reducing the field strength.

3.4.1 Energy of a Capacitor

Consider the same parallel plate schema as before; we know that $C = Q/\phi$. The question becomes, how much energy do we need in order to create such a configuration, with charge $\pm Q$ on either plate?

Suppose we have q already on the top plate, and we wish to add dq from the bottom plate to the top plate. This charge must go through a potential difference of ϕ. The change in energy is given by

$$dU = \phi dq = \frac{q dq}{C} \tag{3.9}$$

Integrating with respect to charge, we see that the total energy required is:

$$W = \frac{1}{C}\int_0^Q q\ dq = \frac{Q^2}{2C} \tag{3.10}$$

Because $Q = C\phi$, the energy U stored in a capacitor can be expressed as

$$U = \frac{1}{2}C\phi^2 \tag{3.11}$$

[2]If that should happen, electrons would jump from the negatively charged plate to the positive one—in other words, lightning!

4 The Electric Current

4.1 Current in a Wire

Moving charges are known as an electric current; an electric current flowing through a wire is like a pipe filled with rushing water. The units of this property is given by the amount of charge moving through a fixed mark in a given second:

$$1 \text{ ampere } = \frac{1 \text{ coulomb}}{1 \text{ second}}. \tag{4.1}$$

When we talk about current, we only consider the net charge that is transported; the reason water doesn't have a current when it flows through a hose is becase equally many protons move as electrons. At any given time, there are electrons flowing through a given cross-sectional area; this gives rise to the notion of current density, which is the amount of charge flowing through an area in unit time. By convention, current can be considered as the flow of positive charge.

$$\mathbf{J} = \rho \mathbf{v} \tag{4.2}$$

Where ρ is the charge density and $\mathbf{v}$ is the velocity of the charge. Note that ρ has units of charge/volume, and $\mathbf{v}$ has units of length/time; the density therefore takes units of charge/area/time. If the charges flowing are discrete (even after approximation), then the current density is simply

$$\mathbf{J} = nq\mathbf{v} \tag{4.3}$$

Where n is the number of charges per unit volume, and $\mathbf{v}$ is the average velocity. If we take a small area $d\mathbf{A}$ at a given place in the material, then the amount of charge flowing through the area in a unit of time is:

$$\mathbf{J} \cdot d\mathbf{A} = \|\mathbf{J}\| \|d\mathbf{A}\| \cos\theta = I \tag{4.4}$$

which is simply the current. If we want to find the current through an arbitrary surface S, we simply take the integral of the density over the entire surface:

$$I = \iint_S \mathbf{J} \cdot d\mathbf{A} \tag{4.5}$$

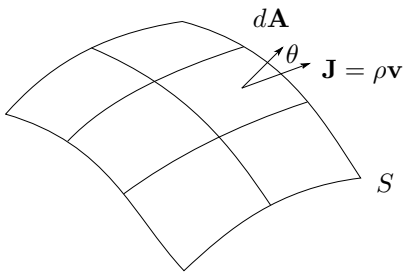

Figure 15: The total current flowing through S is given by the integral of the density over the entire surface.

Of course, the velocity of each individual charge may differ; so in computing the current density we simply consider the average velocity of the charge carriers.

Now let's imagine a volume V that is completely closed off; the integral over this entire surface is the amount of current that enters or leaves the boundary ∂V. Lets also consider a density $\mathbf{J}$. Unlike with electric fields, we are dealing with tangible objects, not mathematical phenomena. Thus we are subject to *conservation of charge*; charge can neither be created nor destroyed. The charges move around, but never appear (or dissapear) from nowhere. Therefore, if current flows out of an enclosure, the charge inside the enclusure must decrease by the corresponding amount. This allows us to make the statement that

$$\iint_{\partial V} \mathbf{J} \cdot d\mathbf{A} = -\frac{dQ}{dt} \tag{4.6}$$

where Q is the charge in V. The charge inside can be written as a volumetric integral:

$$Q = \iiint_V \rho dV. \tag{4.7}$$

If we apply (4.6) to a small volume ΔV, we see that the left handed integral is $\nabla \cdot \mathbf{J}\Delta V$ The charge inside is $\rho\Delta V$, so we can state the conservation of charge as:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{4.8}$$

As the volume ΔV shrinks, we can eventually take ρ out of the volume integral. The integral is carried out in one instant in time, so the time derivative of ρ only depends on the difference between $\rho(\mathbf{x}_0, t)$ ad $\rho(\mathbf{x}_0, t + dt)$, because the position of the volume is infinitely small and held fixed, and therefore its boundary is the same.

4.2 Conductivity and Ohm's Law

An electric field $\mathbf{E}$ pushes (positive) charge carriers in one direction; the result is an electric current in the direction of the field. In most substances, and in a variety of field strengths, we find that the current density is directly proportional to the electric field.

$$\mathbf{J} = \kappa \mathbf{E} \tag{4.9}$$

where κ is a scalar called the *conductiviy* of the material; it is large for conductors and small for insulators. Even though we mentioned that the electric field was 0 inside conductors, it is woth noting that this is no longer a static situation, so an electric field indeed exists within the material.

The above equation is an empirical law known as *Ohm's Law*; it is not true in certain cases. A point of interest is finding the total current I through a wire, and the potential difference V between the two ends. We know that if the density is parallel to the cross sectional area vector, $I \sim \|\mathbf{J}\|A$, where A is the cross-sectional area of the wire. However, we must also note that V is simply the line integral of the field along the length of the wire, so it is also proportional to the current because it is proportional to the field which is proportional to the current density. This gives rise to the relationship

$$V = IR \tag{4.10}$$

where R is known as the *resistance* of a circuit, and is inversely proportional to the conductivity of the material.

In this equation we made some clandestine assumptions; first, we assumed that the current density along the wire is uniform, which can be verified by taking the line integral along a closed loop. Another assumption is that the wire is surrounded by an electrically nonconducting material.

Without this, it is nonsensical to talk about singular currents and singular resistances; we would not know with respect to what we referred, as the charges are free to travel everywhere.

4.3 The Resistor

Electrical devices have well-defined conducting terminals to which conducting wires may be attached. Charge can flow into and out of the devices via these paths, so it makes sense to talk about *the* current and *the* voltage. At these terminals, the charges are equal and opposite, because the currents are steady and the potentials are constant everywhere. So far, we have been dealing with current densities that are invariant with respect to time. This means that they have only one direction, so such currents are known as *direct currents* (or DC).

One such circuit element is called a *resistor*, whose sole purpose is to provide resistance to the current flowing in the wire. There are many ways a certan resistance, measured in Ohms (Ω), can be achieved, as we will later see. Thus, resistance is a more abstract, encapsulating term, so it is denoted by the symbol:

Resistors can be placed *in series* or *in parallel*.

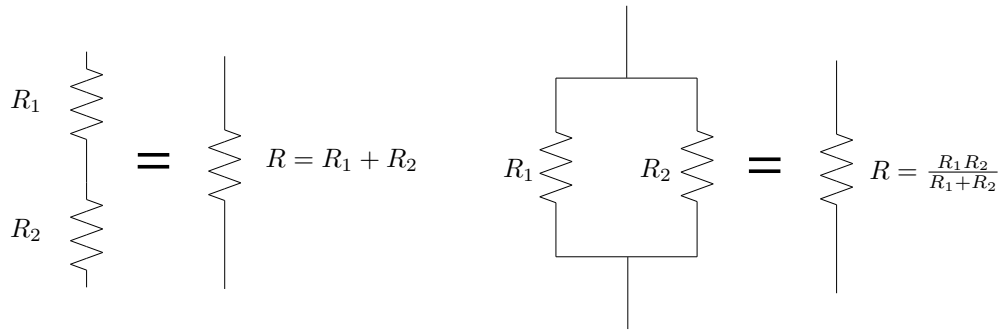

Figure 16: Equivalent resistors for resistors in series and in parallel, respectively.

For these configurations, there are certain laws governing how the resistances combine. For n resistors laid out in series, the total resistance is

$$R_{total} = R_1 + \cdots + R_n \tag{4.11}$$

For resistors placed in parallel, it is

$$\frac{1}{R_{total}} = \frac{1}{R_1} + \cdots + \frac{1}{R_n} \tag{4.12}$$

For every circuit, the following rules must be satisfied:

1. The current through each element must equal the voltage accross that element divided by the resistance ($V = IR$).

2. At each node of the network—a point where three or more wires meet, the currents into the node must sum to zero (equivalent to conservation of charge).

3. The sum of potential differences taken in order around a loop of a circuit network is zero (which is a manifestation of the fact that **E** integrated around a closed loop is zero).

4.4 Variable Currents

Suppose we have a capacitor with capacitance C be charged to a potential V_0, and let it be discharged suddenly by connecting it across a resistance R. Obiously, as current flows, the capacitor will be gradually discharged; in doing so, the voltage difference will diminish, so the current will be lessened. We can start by listing familiar equations:

$$Q = CV_0 \ , \ I = \frac{V}{R} \ , \ I = -\frac{dQ}{dt}. \tag{4.13}$$

Because current is the change of (positive) charge over the change in time, we set it equal to the negative time derivative of charge (positive charge flows out of the capacitor, so the total charge decreases).

Eliminating V and setting the two equations for I equal to each other:

$$\frac{dQ}{dt} = -\frac{Q}{RC} \tag{4.14}$$

we can rearrange the equation:

$$\frac{dQ}{Q} = -\frac{dt}{RC}. \tag{4.15}$$

It is worth noting here that we can check our work so far by looking at the units of RC; indeed, they are units of time. We integrate both sides of the equation to obtain

$$\ln Q = -\frac{t}{RC} + k \tag{4.16}$$

where k is an arbitrary constant. The solution is therefore:

$$Q = e^k e^{-t/RC}. \tag{4.17}$$

The constant term e^k is determined by the inital conditions of the problem; namely, that the capacitor is charged to voltage V_0. The charge behaves like:

$$Q(t) = CV_0 e^{-t/RC} \tag{4.18}$$

and taking the time derivative gives us the current:

$$I(t) = \frac{V_0}{R} e^{-t/RC} \tag{4.19}$$

5 Relativity in Electromagnetic Theory

5.1 Some Brief Context

When Einstein published his seminal paper in 1905 on the principles of Special Relativity, it was not a discovery *ex nihilo*; the likes of Lorentz and others who explored moving charges long before laid the foundations for Einstein's theories.

The basis of any relativistic theory is to demonstrate that the laws governing a physical system are invariant no matter which reference frame is chosen, as long as it is inertial (i.e., with constant velocity). The first of the relativity theories is that of Gallilean Invariance, which postulates that Newton's laws are the same in any inertial frame. And this is true up to a very good approximation; it is why you can drink coffee on an airplane just as easily as you can on the ground, even though you are moving at hundreds of miles an hour.

With this in mind, it does not make any sense to consider "absolute" motion, since it has no consequences on how physical laws determine the evolution of systems. Objects have velocities only relative to one another, and any statement of an object's speed tacitly assumes some kind of an inertial reference frame. As Einstein remarks in the 1905 paper,

> "...[T]he phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the 'Principle of Relativity') to the status of a postulate, and also introduce another postulate...that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell's theory for stationary bodies."

Einstein's principle of relativity asserts that every physical law and fundamental physical constant (including, in particular, the speed of light in vacuum) is the same for all observers in an inertial frame. That is, the light emitting from the headlights of your moving car is not somehow faster. The governing equations that describe electromagnetic interactions are Maxwell's equations; these describe the behavior of magnets, charges, currents, and light, which is a disturbance in the electromagnetic field. However, these equations, when applied to moving bodies, lead to asymmetries which did not appear to be inherent in the phenomena. [NEATEN]

5.2 Lorentz Transformations

Generally speaking, a reference frame is a coordinate system with some specific origin, where the measurement of distance and time are everywhere possible. A stronger condition is that the times measured at different points in the reference frame are synchronized; this is the distinguishing characteristic of a reference frame.

Therfore, any event that occurs is located in space and time by its coordinates x, y, z, and t in some given reference frame. In another reference frame moving relative to it, we denote the coordinates by x', y', z', and t'. Because the speed of light is constant in all inertial reference frames, if we have one obesrver A at (relative) reset and another observer A' with relative velocity $\mathbf{v} = v\hat{x}$, we cannot simply assume that for an event E, $t' = t$. We must remember that A and A' have different clocks, such that the speed of light must be constant for each of them.

We can visualize the setup below:

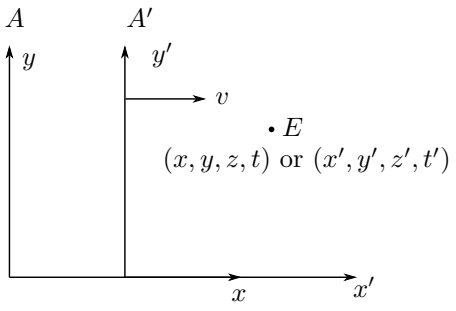

Figure 17: Two reference frames, A and A', moving with velocity v with respect to one another.

Lorentz noticed something interesting when he made the following substitutions in Maxwell's equations:

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned} \tag{5.1}$$

which is that Maxwell's equations remained the same when this transformation was applied to them! As a result, these transformations are known as *Lorentz Transformations*. We define the quantities

$$\begin{aligned} \beta &:= \frac{v}{c} \\ \gamma &:= \frac{1}{\sqrt{1 - \beta^2}} \end{aligned} \tag{5.2}$$

Consider a rod that is stationary in A', parallel to the x-axis. Its length in A' is simply the the difference between its endpoints, or $x'_2 - x'_1$. However, in the frame A, we see that its length is

$$x_2 - x_1 = \frac{x'_2 - x'_1}{\gamma}. \tag{5.3}$$

Keep in mind the limiting cases; as the velocity v of A' goes to zero, the lengths become equal. When $v \to c$, $\gamma \to \infty$, so the rod's length in A becomes zero. Simply stated, the lengths of objects are squished depending on how fast one moves. This is the famous *Lorentz Contraction*. Length perpendicular to the velocity's direction is measured to be the same.

In a similar manner as length, time also feels these effects, and can "dilate" as necessary. Suppose we have a clock in A' moving with speed v, and let us record t'_1 as it passes one clock in the frame A, which records time t_1. Later, the moving clock passes another clock in A, and when this occurs,

the time t_2' is recorded in the moving reference frame and t_2 in the stationary one. We see that in the Lorentz transformation,

$$t_2' - t_1' = \frac{t_2 - t_1}{\gamma}. \tag{5.4}$$

Again, in the limiting case, as $v \to 0$, $\gamma \to 1$, so the times are in agreement. However, as $v \to c$, $\gamma \to \infty$, so the clocks run infinitely slowly; no time has elapsed for the frame moving at the speed of light. Remember that moving clocks move slower.

5.3 Velocity, Momentum, and Energy

Clearly, the model of vector-additive velocities needs some touching up. For instance, suppose an object is moving with velocity u_x in the positive x direction in our moving frame A'. What is its velocity in the stationary one?

By taking x/t, we obtain

$$u_x = \frac{u_x' + v}{1 + u_x' v/c^2} \tag{5.5}$$

Now consider a paricle moving at velocity $\mathbf{v}$ in an inertial frame A. We find that energy and momentum are conserved in the interactions if we make momentum and energy as such:

$$\begin{aligned} \mathbf{p} &= \gamma m_0 \mathbf{v} \\ E &= \gamma m_0 c^2 \end{aligned} \tag{5.6}$$

Where m_0 is the mass of the particle taken at rest.

5.4 Interaction Between Electric Current and Moving Charge

Suppose you have the following setup. A wire is carrying a current I which flows in a particular direction (recall that by convention, current is defined to be the flow of positive charge). This wire is electrically neutral, so it has no net charge density. If we wanted to be even more precise, we would say that it is electrically neutral *in a given frame*, which we call the "lab frame" or frame A.

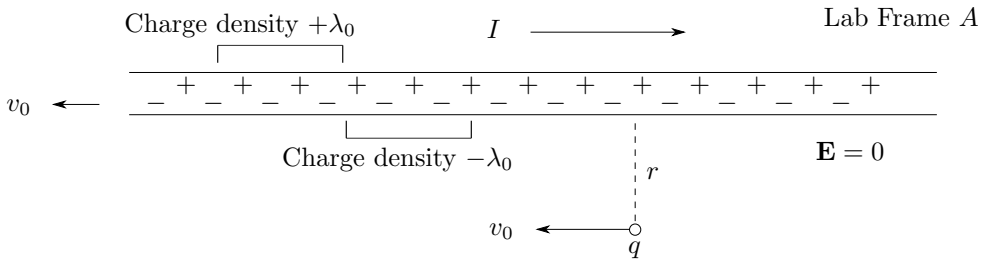

Figure 18: Lab frame A. Electrons moving with speed v_0, charge density $-\lambda_0$, positive charges at rest with charge density $+\lambda_0$. A test charge q moves parallel to the current in a wire with speed v_0.

Suppose our test charge were stationary; what would we expect to happen? Because the wire is electrically neutral, we can refer to the formula for a linear charge density, and find that the field everywhere due to the wire is zero. It might be tempting to say that a moving charge would not experience any force either—but this is not the case! When the charge is moving, we observe that

there is in fact a force that it experiences. All we must do is consider the reference frame of the moving test charge, A'.

Because the reference frame A' is moving with the same speed and direction as the electrons, the electrons are at rest, as well as the test charge. The positive charges are moving in the other direction at speed v_0, and the current is still I. (How come?) We know from our previous discussions that because the positive charges are moving, there must be a length contraction in the direction parallel to the travelling positive charges, which changes the charge density. Moreover, because the electrons were moving and had a density of $-\lambda_0$, they must have less density in the frame in which they are at rest.

We can think of the positive charges as being laid along the ticks of a ruler; when the ruler moves, it is length contracted and the ticks come closer together.

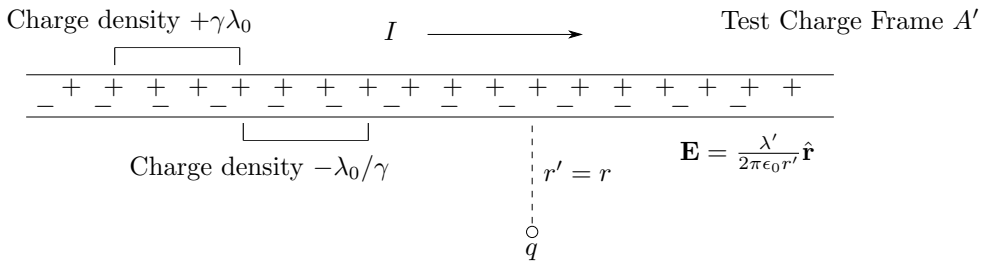

Figure 19: Test charge frame A'. Here, the electrons are at rest, with corresponding charge density $-\lambda_0/\gamma$, the protons have charge density $+\gamma\lambda_0$, and the test charge is statoinary. This gives rise to an electric field.

Therefore, because the electrons are farther apart and the positive charges are closer together, this means that there is a net positive charge density, so the test charge does experience a force! Namely, the wire repels the positive test charge in the downward direction.

This is the crucial relationship between Special Relativity and Electromagnetism. For a moving charge, there must *necessarily* be a force on a moving particle and a field in the reference frame A'. We can calculate this force. We first set

$$\begin{aligned} \beta &= \frac{v_0}{c} \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}} \end{aligned} \tag{5.7}$$

The new linear charge density is now

$$\lambda' = \gamma\lambda_0 - \frac{\lambda_0}{\gamma} \tag{5.8}$$

We also know that the equation of the electric field due to a linear charge is

$$E' = \frac{\lambda'}{2\pi\epsilon_0 r'} = \frac{\lambda_0(\gamma - 1/\gamma)}{2\pi\epsilon_0 r} \tag{5.9}$$

We know that $r' = r$ because lengths are not contracted in the direction perpendicular to the velocity. By a little algebra, we see that

$$\begin{aligned} E' &= \frac{\lambda_0(\gamma - 1/\gamma)}{2\pi\epsilon_0 r} \\ &= \frac{\lambda_0\left(1/\sqrt{1-\beta^2} - \sqrt{1-\beta^2}\right)}{2\pi\epsilon_0 r} \\ &= \frac{\lambda_0(1-(1-\beta^2))}{2\pi\epsilon_0 r\sqrt{1-\beta^2}} \\ &= \frac{\lambda_0 v_0^2}{2\pi\epsilon_0 r c^2\sqrt{1-\beta^2}} \\ &= \frac{\lambda_0 v_0^2\gamma}{2\pi\epsilon_0 r c^2}. \end{aligned} \tag{5.10}$$

Therfore, the test charge will experience a downward force

$$F' = qE' = \frac{q\lambda_0 v_0^2\gamma}{2\pi\epsilon_0 r c^2} \tag{5.11}$$

If we return to the lab frame, observers will report a smaller force, given by

$$F = \frac{F'}{\gamma} = \frac{q\lambda_0 v_0^2}{2\pi\epsilon_0 r c^2} \tag{5.12}$$

But recall that $v_0\lambda_0$ is just the current flowing through the wire. We can finally write the equation

$$F = \frac{qv_0 I}{2\pi\epsilon_0 r c^2} \tag{5.13}$$

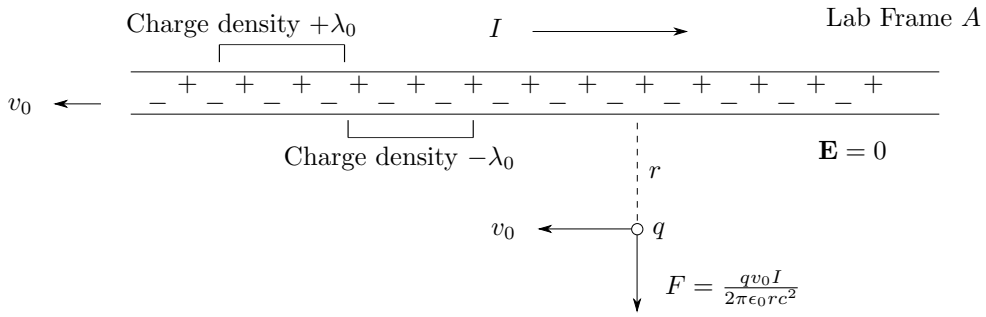

Figure 20: The test charge experiences a magnetic force.

We call this force which depends on the velocity. We will soon introduce a vector field $\mathbf{B}$ called the *magnetic field*, such that $q\mathbf{v}\times\mathbf{B}$ will always give the direction and magnitude of the magnetic force.

Notice how in frame A, the test charge is subject to a magnetic force, whereas in frame A', the test charge experiences an electric force. Therefore, if there an electric force in frame A' is the magnetic force in frame A', and vice versa.

In general, a current-carrying wire will not be electrically neutral in all inertial frames. Once a wire is carrying currents, there is a velocity difference between the positive and negative charges, so there is a difference in how they transform into another frame. Therefore, there will be a net charge in the other frame.

These are the obseservations that led the likes of Lorentz and Henri Poincaré to build the structures that led Einstein to his theories.

6 Magnetostatics

A charge that is moving parallel to a current of other charges experiences a force perpendicular to its own velocity. At some instant t a particle with charge q passes the point (x, y, z) in our frame, moving at velocity $\mathbf{v}$. The total force on this particle is known to be $\mathbf{F}$, and the electric field is $\mathbf{E}$. Therefore, we define the magnetic field to be the vector $\mathbf{B}$ such that it satisfies the following equation for any $\mathbf{v}$:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{6.1}$$

This force $\mathbf{F}$ is known as the Lorentz force, which is essentially the force on a test particle q. We can take this Lorentz force to be *the* definition of $\mathbf{E}$ and $\mathbf{B}$. Notice how when we introduced the electric field $\mathbf{E}$, we defined it as the force per unit charge. Now we have just generalized its definition to include movement. One can imagine finding the electric field locally by first holding a charge fixed to obtain the electric field, and then moving it around and observing how the force changes to determine the magnetic field.

The implications of this, therefore, is that the additive portion of the Lorentz force law, the units of $\mathbf{B}$ are units of electric field/velocity.

We define the constant

$$\mu_0 := 4\pi \times 10^{-7} \frac{\text{kg m}}{\text{C}^2} \tag{6.2}$$

Which gives rise to the following equality:

$$c^2 = \frac{1}{\mu_0 \epsilon_0} \tag{6.3}$$

Therefore, the relationship between the the field and current in [INSERT EQUATION] just becomes

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\mathbf{z}} \tag{6.4}$$

More generally, we can calculate the force on a small piece of current-carrying wire that sits in a magnetic field $\mathbf{B}$:

$$d\mathbf{F} = I d\boldsymbol{\ell} \times \mathbf{B} \tag{6.5}$$

If the magnetic field can only be caused by a current, then you might wonder how ordinary magnets like the ones on your refrigerator work. Where is the electric current? As it happens, the macroscopic magnetic field in some magnetic materials is produced by the alignment of tiny atomic currents generated in the material.

6.1 The Right-Hand Rule

Many of you may be familiar with the *cross product*, denoted $\times$, which is a way to compute a vector perpendicular to two given vectors:

$$\mathbf{x} \times \mathbf{y} = \mathbf{z} \tag{6.6}$$

If the two vectors are parallel, then the cross product is zero. If they are not parallel, then we can use the right-hand rule to find out the direction where it points. In order to compute the direction of a cross product, $\mathbf{x} \times \mathbf{y}$:

1. Put the fingers of your right hand in the direction of $\mathbf{x}$.

2. Curl your fingers in the direction of $\mathbf{y}$, so that they sweep along the plane containing $\mathbf{x}$ and $\mathbf{y}$.
3. Keeping your fingers curled and your hand motionless, extend your thumb in a "thumbs-up" gesture. Your thumb will point in the direction of $\mathbf{x} \times \mathbf{y}$.

Notice what happens when you reverse the order of the cross product and take $\mathbf{y} \times \mathbf{x}$; Your fingers cannot curl that way, so you are forced to flip your hand over. This leads us to the familiar result of

$$\mathbf{x} \times \mathbf{y} = -(\mathbf{y} \times \mathbf{x}) \tag{6.7}$$

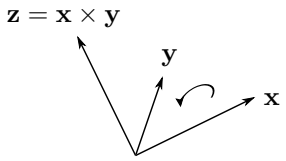

Figure 21: Put the fingers of your right hand along $\mathbf{x}$, then curl them in the direction of $\mathbf{y}$. Your thumb will point in the direction of $\mathbf{x} \times \mathbf{y}$.

There are other types of right-hand rules that are very useful. For instance, we considered the simple configuration of a wire carrying a current I. Because a charge moving parallel to the wire experiences a downward force, we have to fid a third vector $\mathbf{B}$ that is perpendicular in order to satisfy

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}, \tag{6.8}$$

which is the Lorentz force law in no electric field. Because of rotational symmetry, one can see that the magnetic field vectors form rings around the wire, that wrap around and are always perpendicular to the radius. The question is, do they wrap around the wire clockwise or counterclockwise?

By convention, one uses the right-hand rule to find the direction of the magnetic field wrapping around a wire. To do this, point your thumb in the direction of the current flowing, and your fingers will curl in the orientation of the magnetic field.

The same works in the other direction; suppose you have a circular wire of current. In order to find the magnetic field, you curl your fingers in the direction of the flowing current, and your thumb will point in the direction of the magnetic field, which goes through the loop[3].

You might ask why the universe cares so much about your right hand. It doesn't. Handedness of computations is just a convention; we easily could have used the "left-hand rule," and, as long as we were consistent in using it, our predictions wouldn't change. This is because there is a right hand rule involved in determining the $\mathbf{B}$ field, and another in determining $\mathbf{v} \times \mathbf{B}$. These effects cancel, and yield the same predictions. Because the dominant hand of most people is the right one, humans have adopted the right-handed convention.

6.2 Ampère's Law

The magnetic field, like the electric field, is a device for describing how charged particles interact. Recall from our earlier discussion on current that

[3]Really, your thumb points in the direction of the magnetic field in the exact center of the loop, because the field skews slightly the further out from the center you go.

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

If the current is steady (i.e. constant with time), this term goes to zero, which is the condition in *magnetostatics*:

$$\nabla \cdot \mathbf{J} = 0. \tag{6.9}$$

Like we discussed earlier, because the magnetic field curls around the wire, taking the line integral around a closed loop will *not* yield zero! This is because the more we turn around the wire, we accumulate more in our integral; we can make it so that a closed loop never has a negative dot product with a differential arc length. The fact that the magnetic field has nonzero curl—or in other words, is not conservative—means we can define a quantity for this line integral around a closed loop. We claim that the value obtained by integrating around a path that encloses a wire is the same, and we state the following law, known as Ampère's law:

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I \tag{6.10}$$

where $\mathbf{B}$ is the magnetic field, $d\boldsymbol{\ell}$ is the differential arc length, and I is the current enclosed by the loop surrounding the wire.

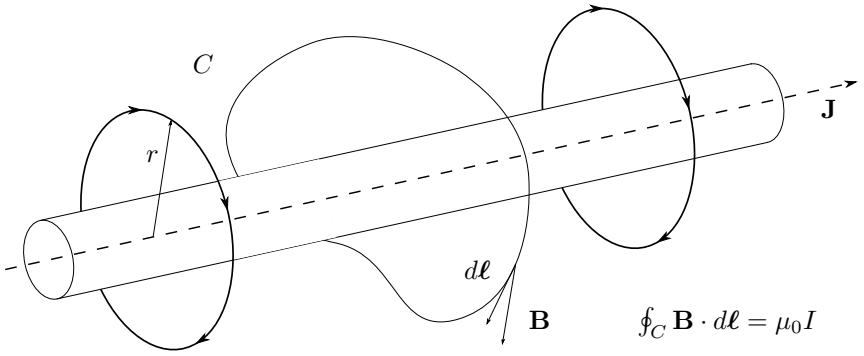

Figure 22: Ampère's law tells us that the line integral around any closed curve C equals the total current enclosed by the curve.

Of course, for the specific example of the field due to a wire with steady current, it makes the most sense for our path to be a circle such that the magnetic field is everywhere parallel around the wire. Therefore, the magnetic field of a long, straight wire with current I is just

$$\begin{aligned} \oint_C \mathbf{B} \cdot d\boldsymbol{\ell} &= \mu_0 I \\ \|\mathbf{B}\| \oint d\boldsymbol{\ell} &= \mu_0 I \\ \|\mathbf{B}\|(2\pi r) &= \mu_0 I \\ \|\mathbf{B}\| &= \frac{\mu_0 I}{2\pi r} \end{aligned} \tag{6.11}$$

We can rewrite Ampère's law as

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{A} \tag{6.12}$$

Remember due to Stokes' theorem,

$$\oint_C \mathbf{F} \cdot d\boldsymbol{\ell} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A} \tag{6.13}$$

where $\nabla \times \mathbf{F}$ is the curl operation for the vector field $\mathbf{F}$. We see that an equivalent statement to Ampère's law then becomes

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{6.14}$$

This is known as the differential form of Ampère's law for currents that do not change in time.

We must also remark that the divergence of the magnetostatic field is zero; this is because we cannot make a tiny volume such that there is a net inward or outward flux, even if we enclose the wire. If this is true for the field of a wire, then by superposition we assert that

$$\nabla \cdot \mathbf{B} = 0 \tag{6.15}$$

for all time-invariant magnetic fields.

6.3 The Vector Potential

Recall that the scalar potential function $\phi(\mathbf{r})$ gave us a simpler way to calculate the field of a static charge distribution. We used the property that the $\mathbf{E}$ field was conservative to define obtain the relationship

$$\mathbf{E} = -\nabla\phi. \tag{6.16}$$

However, this won't exactly work here; we cannot assume that the magnetic field is conservative, and as a matter of fact, it is not. If you remember anything from vector calculus, it is that the curl of a function yields a vector field, and that field is divergence-free. Moreover, it is worth noting that gradient vector fields are curl-free, so that is why the electric field does not curl.

Symmetry considerations led us to believe that the divergence of the magnetic field is precisely zero; therefore, we are guaranteed a solution if we can find a vector field $\mathbf{A}$ such that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6.17}$$

and we can automatically say that the divergence of the magnetic field is zero. Such a function $\mathbf{A}$ is the *vector potential*. From the differential form of Ampère's law, we see that

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} \tag{6.18}$$

We can introduce one of the standard identities in vector calculus, namely that

$$\nabla \times (\nabla \times \mathbf{F}) = -\nabla^2 \mathbf{F} + \nabla(\nabla \cdot \mathbf{F}) \tag{6.19}$$

We have some freedom in how we construct $\mathbf{A}$. All that it needs to satisfy is that its curl is the magnetic field; the divergence can be set to zero without changing anything about that. Therefore, the previous equation simplifies:

$$-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \tag{6.20}$$

Which is a familiar equation from our discussion on Poisson's equation. Replacing ρ/ϵ_0 with $\mu_0 \mathbf{J}$, we get the vector potential equation:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r'})}{\|\mathbf{r'} - \mathbf{r}\|} dV \tag{6.21}$$

6.4 Magnetic Monopoles and Aharonov-Bohm Effect

We took the equation $\nabla \cdot \mathbf{B} = 0$ to be true, but we never thought about what exactly it is telling us. If we take positive divergence to mean there is a "source" and a negative divergence to be a "sink," then this equation has a simple interpretation: there are no magnetic charges. There is no box you can make such that the net flux of the magnetic field is greater than zero. Experimentally, the search for such sources of magnetic field are ongoing, yet no magnetic monopole has been found. However, despite never having observed a magnetic charge or "monopole," we have sound reasons for believing they do indeed exist.

These monopoles are of particular interest today, because modern theories predict their existence. The work of 't Hooft and Polyakov has shown that the existence of monopoles is a conclusion of very general ideas about the unification of fundamental interactions. Of course, having a nonzero divergence means that we will no longer be able to express the magnetic field as the curl of another vector potential $\mathbf{A}$.

However, there is a problem with this. The 1959 Aharonov-Bohm experiment showed that charged particles are affected by an electromagnetic potential $(\phi, \mathbf{A})$ despite being in regions where the corresponding fields $(\mathbf{E}, \mathbf{B})$ are zero. This suggests that there is some physical significance to the potential beyond being an abstract mathematical tool to compute fields.

The legendary physicist Paul A.M. Dirac showed that it is actually possible to construct a vector potential $\mathbf{A}$ which allows for the presence of magnetic charges, but only if the magnetic charge g is related to the charge of the electron e by the Dirac quantization condition:

$$ge = 2\pi\hbar n, \text{ for } n \in \mathbb{Z} \tag{6.22}$$

where $\hbar$ is the reduced Planck Constant. That is, the strength of the monopoles must occur in integer multiples related to the electric charge. We previously observed that the charge of an electron does come in these little packets, called "quantizations"—but we never had a reason for this. One argument is that these quantizations come from the existence of some potential function $\mathbf{A}$.

Unfortunately, the chance that we find monopoles are exceedingly slim. As of now, they are beyond the capabilities of our particle accelerators to produce. The search still continues.

6.5 The Biot-Savart Law

We can find the field from any current-containing wire by using superposition to add the differential lengths in wire. For a thin wire, we can take the differential volume element dV to be a short section $d\ell$. Then, we replace $\mathbf{J}dV$ with $Id\boldsymbol{\ell}$. We can write the vector potential equation as a line integral over the entire wire, or the curve C:

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \int_C \frac{d\boldsymbol{\ell}}{r} \tag{6.23}$$

where r is the distance from the point we are evaluating from the diferrential arc length. In differential terms, we can obtain

$$d\mathbf{A} = \frac{\mu_0 I}{4\pi} \frac{d\boldsymbol{\ell}}{r} \tag{6.24}$$

Taking the curl of $d\mathbf{A}$ gives us $d\mathbf{B}$ by definition. Therefore,

$$\begin{aligned} d\mathbf{B} &= \nabla \times \frac{\mu_0 I}{4\pi} \frac{d\boldsymbol{\ell}}{r} \\ &= \frac{\mu_0 I}{4\pi} \left(\frac{1}{r} \nabla \times d\boldsymbol{\ell} + \nabla \left(\frac{1}{r} \right) \times d\boldsymbol{\ell} \right) \\ &= \frac{\mu_0 I}{4\pi} \left(-\frac{\hat{\mathbf{r}}}{r^2} \times d\boldsymbol{\ell} \right) \end{aligned} \tag{6.25}$$

The resultant equation is known as the Biot-Savart Law:

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{d\boldsymbol{\ell} \times \hat{\mathbf{r}}}{r^2} \tag{6.26}$$

Note that the minus sign disappears because we switch the order of the cross product, because the cross product is anticommutative.

6.6 Examples

6.6.1 Solenoid

Consider the following construction of a wire that loops around many times, and stacks on itself like a coil (think of a long slinky). By the right hand rule, the magnetic field contributions from each of the N turns of the wire are all in the same direction, because the current is flowing through each turn in the same direction. This construction is called a solenoid, and is taken to be very long, with $h \gg r$.

Inside the solenoid, the magnetic field is essentially uniform and pointing in only one direction. Therefore, we can construct a curve that is rectangular, such that the top side of length a is parallel to the magnetic field, the bottom lies outside the coil, and the other two sides are perpendicular to the magnetic field.

Therefore, because the field is zero on the outside, and the other two sides are perpendicular, we see that the curve we constructed only has a contribution along the one side. Moreover, assuming that the coils are uniformly distributed, we see that the proportion of the coils in our loop is simply

$$I' = IN\frac{a}{h} \tag{6.27}$$

Therefore,

$$\begin{aligned} \oint_C \mathbf{B} \cdot d\boldsymbol{\ell} &= \mu_0 I' \\ \|\mathbf{B}\| \cdot a &= \mu_0 I' \\ &= \mu_0 IN \frac{a}{h} \end{aligned} \tag{6.28}$$

Therefore, for a finite solenoid, the interior magnetic field is just

$$\mathbf{B} = \frac{\mu_0 IN}{h} (\hat{\mathbf{r}} \times \hat{\mathbf{J}}) \tag{6.29}$$

Where $\hat{\mathbf{r}}$ is the radial component inside the solenoid, and $\hat{\mathbf{J}}$ is the direction of the current density in the coils. Because these two are always perpendicular, we obtain the third perpendicular unit vector with the correct sign.

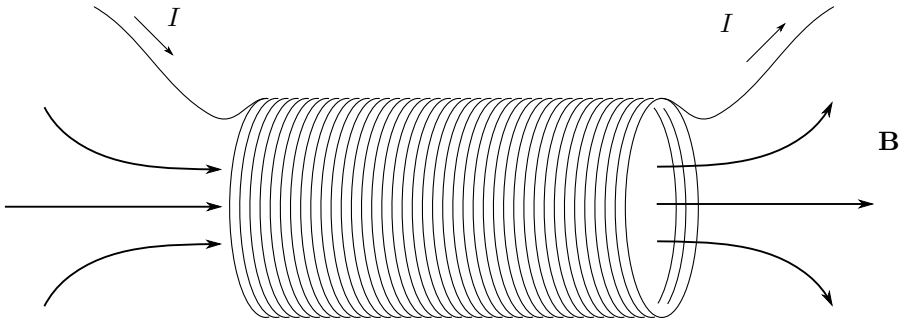

Figure 23: Magnetic field in a solenoid. Near the middle, the field is almost completely uniform

7 Electromagnetic Induction

7.1 Conducting Rod in a Uniform Magnetic Field

Suppose we have a conducting rod in a uniform magnetic field and no electric field. The force on the particles in the field are therefore

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) \tag{7.1}$$

As we move the rod, the positive and negative charges experience a force, and move to opposite ends of the rod (technically, only electrons are free to move in conductors, but the effect is still the same).

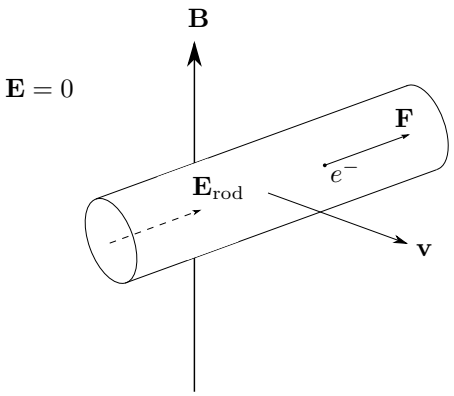

Figure 24: The electrons in the conducting rod (e^-) experience a force $\mathbf{F} = -e(\mathbf{v} \times \mathbf{B})$. Therefore, there is a potential difference that arises in the rod and the rod is polarized.

Now because we have charges on either side of the rod, there is now a potential difference between the two sides, and therefore an electric field created in the rod.

This seemingly contradicts what we covered in Section 3, where we spent a great deal of time demonstrating that conductors are equopotentials and do not have a field within them. However, that was only in the static case. When the conductors move in a magnetic field, they are no longer equipotential; we will later see this is also the case in a changing magnetic field.

7.2 Conducting Loop in a Nonuniform Magnetic Field

Suppose we have a closed loop of wire that is approximately a square, and we start it in a region with no magnetic field, and then drag it into a region with a magnetic field. What happens to the charges inside?

Like before, the charges go to either end of the loop that is inside the magnetic field. However, because part of the loop does not have any magnetic field to counteract, we can induce a current, becuase the charges move around.

As the loop is pulled through, note that it makes a magnetic field pointing in the direction opposite of $\mathbf{B}_{\text{in}}$. This is a premonition about how inductors work, as they try to counteract the magnetic field. We are now ready to generalize our findings.

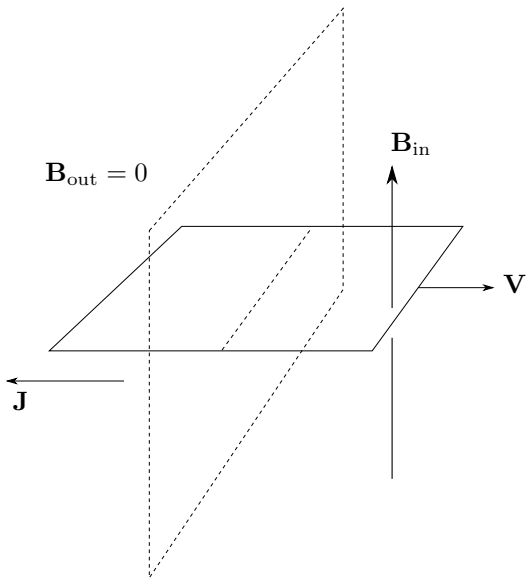

Figure 25: We can induce a current through a loop by dragging it through a nonuniform magnetic field.

7.3 Universal Law Of Induction

Suppose we create have a surface S that is bounded by ∂S. First, we define the magnetic flux as

$$\Phi = \iint_S \mathbf{B} \cdot d\mathbf{A} \tag{7.2}$$

and we denote the time derivative of this flux as

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{7.3}$$

where $\mathcal{E}$ is the electromotive force.

When this occurs, there is a current induced along the boundary ∂S corresponding to the right-hand rule. Due to the current, we have an electric field. We can write Faraday's law:

$$\mathcal{E} = \oint_{\partial S} \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot d\mathbf{A} = \frac{d\Phi}{dt} \tag{7.4}$$

7.4 Induction and Lenz's Law

Suppose we have the same setup in Figure 25. In what sense does the magnitude of the magnetic field increse as we pull the loop forward into the region?

We can consider the conducting loop ∂S to have a resistance R and to enclose a surface S; by Faraday's law, we see that the area through which the magnetic field flows increases as the loop moves more and more inwards.

The sense in which the line integral and the direction of the flux is determined by the right-hand rule. In the schematic we have written above, the magnetic field is pointing upwards. If we say

the magnetic field is increasing with time, that means we have implicitly defined an orientation for the boundary of the loop, which is related to the direction of increasing flux by the right-hand rule. Therefore, the curve is oriented in the direction *opposite* the flowing current.

Notice that the units of the integral

$$\oint_{\partial S} \mathbf{E} \cdot d\boldsymbol{\ell}$$

are units of Volts. Therefore, the more the loop is dragged into the field, the more negative this integral becomes. We therefore create a negative voltage and a current moving in the direction opposite of the orientation. This in turn creates a magnetic field antiparallel to $\mathbf{B}_{\text{in}}$, to try and counteract it.

This is the most important idea of induction: if you try to change the magnetic field in a loop (or circuit), the loop will run a current to try and counteract that change.

This is a negative feedback relation, which leads to the stability of electromagnetic forces; if not, then we would have an infinitely increasing magnetic field.

As with other important equations, we can specify a differential form by using one of the many integral theorems. By Stokes' theorem, we have

$$\oint_{\partial S} \mathbf{E} \cdot d\boldsymbol{\ell} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{A}. \tag{7.5}$$

Notice that the electric field can no longer be described by the gradient of some potential; since the magnetic field is changing, there is nonzero curl to the electric field. Per equation (7.4), we see that

$$\iint_S \nabla \times \mathbf{E} \cdot d\mathbf{A} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot d\mathbf{A}. \tag{7.6}$$

Therefore, we conclude that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.7}$$

7.5 Mutual Inductance

Recall the setup we had with the solenoid, and recall the equation for the uniform magnetic field within it. Suppose the solenoid is of radius R, and inside of it we place another conducting loop of radius r such that is oriented like the coils. [DRAW]

We can compute the magnetic flux through the loop; since the area is parallel to the direction of the magnetic field, we have that

$$\begin{aligned} \Phi &= \pi r^2 \cdot \|\mathbf{B}\| \\ &= \frac{\pi r^2 \mu_0 I_1 N}{h} \end{aligned} \tag{7.8}$$

If the current I_1 that flows through the solenoid changes with time, then by Faraday's law there is an induced curent I_2 through the loop which seeks to counteract the change. The electromagnetic force is:

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\pi r^2 \mu_0 N}{h} \left(\frac{dI_1}{dt} \right) \tag{7.9}$$

Therefore the voltage in the smaller loop depends on the time derivative of the current I_1.

This is the basic principle of mutual inductance. Notice that due to symmetry, the mutual inductance the solenoid has on the loop is the same as the mutual inductance the loop has on the solenoid. [RECIPROCITY THEOREM?]

7.6 Self-Inductance

We consider the same schematic as last time, with the loop in the solenoid. This ring feels an electromagnetic force as the magnetic field changes; however, this is also true for any coil in the solenoid, who is of radius R:

$$\Phi = \frac{\pi R^2 \mu_0 I_1 N}{h} \tag{7.10}$$

Therefore, by superposition, the total magnetic flux due to all the coils is

$$\Phi_{total} = \frac{\pi R^2 \mu_0 I_1 N^2}{h} \tag{7.11}$$

Therererefore, trying to change the current induces an electromagnetic force;

$$\mathcal{E} = -\frac{\pi R^2 \mu_0 N^2}{h} \left(\frac{dI_1}{dt} \right) \tag{7.12}$$

We can define the quantity

$$L := \mu_0 \left(\frac{N}{h} \right)^2 \pi R^2 h \tag{7.13}$$

Where the latter terms are the volume of the solenoid, and the squared term is a the density of turns per unit length. Notice again that if we attempt to change the current, we get an induced current I_2 that tries to counteract it.

7.7 Circuit Elements

So far, we have covered three major circuit elements: the capacitor, the resistor, and the inductor. We can relate the voltage across each of them to the behavior of the electric current:

Element	V
DC Battery	V
Capacitor	$\frac{1}{C} \int I dt$
Resistor	RI
Inductor	$-L \frac{dI}{dt}$

(7.14)

The astute reader might begin to see how we can construct a differential equation that describes how the voltage changes over time.

7.8 Hello World LC Circuit

We construct a circuit that consists only of a capacitor and an inductor in series. This is the basis for a radio antenna, as we will later see.

If the capacitor has no net charge on either plate, then nothing happens. If we accumulate a positive charge on the top plate, a current will arise as the positive charges flow to the other plate; however, due to the inductor, there is some interesting dynamics that arise. From the voltage law (i.e., the total voltage around the circuit must be zero), we can write

$$\frac{Q}{C} - L\frac{dI}{dt} = 0 \tag{7.15}$$

We have assumed, in this circuit, that there is a small resistance. In physics, when we say a quantity is "small," we would be remiss if we did not say with respect to what it is small. However, we will not discuss relative smallness without digressing to discuss differential equations to the RLC circuit. It may not look like it, but we have actually specified a homogeneous differential equation.

We specify the initial condition of the charge on the capacitor:

$$Q(0) = Q_0 \tag{7.16}$$

When the charge leaves the top plate of the capacitor, we create a current; recall then that the current is simply the negative derivative of charge with respect to time:

$$I = -\frac{dQ}{dt} \tag{7.17}$$

Note that the sign on this equation is set by the sign conventions we have chosen for our circuit. Taking the derivative of the current, we obtain

$$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \tag{7.18}$$

Our previous equation better resembles a differential equation:

$$\frac{1}{C}Q + L\frac{d^2Q}{dt^2} = 0 \tag{7.19}$$

Which we can convert to

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q \tag{7.20}$$

This is a linear second-order homogeneous differential equation. This is a fancy way of saying that we only add scalar multiples of derivative terms, we see a second derivative, and all terms involve Q. Linear homogeneous equations of any order are always solved with exponentials.

If you hearken back to a first course in mechanics, this LC circuit is presicely a harmonic oscillator. We can see this in the general solution to the differential equation:

$$Q(t) = Q_0 \cos(\omega t + \phi) \tag{7.21}$$

Where ω is some parameter with units radians/second, and ϕ is a dimensionless phase angle, which is an arbitrary constant (remember that the "dimension" of radians is not a dimension at all, just a tag to remind us that we are working with angles). From the dimensions of ω, we see that it is a frequency of sorts; we mentioned before that this is an oscillator, so ω is its freqency. You should check that this equation for $Q(t)$ indeed solves the differential equation if and only if

$$\omega^2 = \frac{1}{LC} \tag{7.22}$$

You should also verify that the units of $1/LC$ are $1/\text{seconds}^2$.

It is worth thinking about why the oscillations are generated physically; the charge will flow to the bottom plate, and as that happens, there will be a changing second derivative on the current; then once the charge has accumulated to a certain amount, it will go back to the top plate. This process will continue ad infinitum.

7.9 Hello World RLC Circuits

All circuits are RLC circuits.

This is a powerful fundamental fact of circuitry. If you think about it, all wires technically have some finite resistance. Moreover, a voltage source (like a DC battery) has a potential difference between the top and bottom; this means that there are fields pointing in the direction of decreasing potential, and therefore there must be charges accumulating at the top and the bottom. Finally, because there is a current flowing through the wire, by the right-hand rule there is a magnetic field, which provides inductance. The differential equations that follow are, in that sense, inherent to all circuits that have ever been created.

Before, we considered a highly idealized LC circuit with zero resistance. In that case, the circuit would resonate back and forth, exchanging charge from one side of the capacitor plate to the other. However, with the introduction of resistance, this is not the case, and some energy is dissipated which ensures the oscillation will not continue forever.

To understand this, we must first talk about power in a resistor.

7.9.1 Power in a Resistor

Recall the units of the voltage

$$[V] = \text{volts} = \frac{\text{Joules}}{\text{Coulomb}} \tag{7.23}$$

and the units of the current

$$[I] = \text{amps} = \frac{\text{Coulombs}}{\text{seconds}}. \tag{7.24}$$

Therefore, multiplying them together, we get units of Joules/seconds, which as you may recall are units of power.

$$P = IV \tag{7.25}$$

Therefore, using Ohm's law, we get the power dissipated across the resistor

$$P = -I^2 R \tag{7.26}$$

In other words, resistors take power out of the circuit. We make the sign negative as a convention to indicate that it dissipates power. The units of P are in watts, which are defined to be Joules/second in SI units. When current flows through the resistor, the resistor will dissipate the energy as heat

Having covered this, we can now modify our differential equation to account for the (small) resistance:

$$\frac{Q}{C} - L\frac{dI}{dt} - IR = 0 \tag{7.27}$$

We notice that the current is simply the rate of change of charge. This equation then becomes

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = 0 \tag{7.28}$$

You should verify that the following equation is indeed a solution by taking derivatives and seeing if the equations match.

$$Q(t) = Q_0 e^{-\gamma t/2} \cos(\omega t + \phi) \tag{7.29}$$

The inclusion of this factor of $e^{-\gamma t/2}$ comes from the resistor. You should see that this is satisfied if and only if

$$\begin{aligned} \gamma &= \frac{R}{L} \\ \omega^2 &= \frac{1}{LC} - \frac{R^2}{4L^2} \end{aligned} \tag{7.30}$$

8 Gauge Freedoms, Transformations, and Symmetries

A bird flies down and sits atop a high-voltage wire. Miraculously, it still lives—why? Well, there is no absolute sense in which the volgage of the wire is high. It may as well be one volt or one-hundred million volts; it is only the differences in potential that matter. The voltage is only high in comparison to the voltage of the Earth. We usually define the potential of the "ground" to be 0, but because it is only the potential difference that determines electrodynamic interactions, any other choice would do.

This is a very simple example of a *gauge symmetry* or a *gauge freedom*. It is not a property of nature, but rather a property of how we choose to describe nature. It's a manifestation of the redundancy of our physical models; we have infinitely many valid descriptions of a physical system without altering its laws.

One example of a gauge freedom is coordinate freedom. In Newtonian Mechanics, if you are trying to predict some observable, your result does not depend on the system of coordinates you use. In effect, we have a freedom to choose a coordinate system, and the laws of physics are unaffected by that choice. Moreover, consider the Lorentz force equation for a charged particle:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{8.1}$$

Equation (8.1) relates a vector ($\mathbf{F}$) to a scalar (q), two vectors ($\mathbf{E}, \mathbf{v}$), and a pseudovector ($\mathbf{B}$—see the following section for a description of pseudovectors). These objects—vectors, scalars, and pseudovectors—have certain properties that are invariant with respect to changes of coordinates.

There are more examples of gauge freedoms. For instance, our choice of the vector potential $\mathbf{A}$ in Section 6.3 is far from unique. Recall that gradients must be curl-free. This means we can add any vector potential of the form $\nabla\chi$ and still have the same magnetic field:

$$\mathbf{A}' = \mathbf{A} + \nabla\chi \Rightarrow \nabla \times \mathbf{A}' = \nabla \times \mathbf{A} \tag{8.2}$$

Such a transformation of $\mathbf{A}$ is known as a gauge transformation. By making a judicious choice for χ, it is possible to greatly reduce a problem. We express gauge transformations thusly;

$$\mathbf{A} \mapsto \mathbf{A} + \nabla\chi \tag{8.3}$$

Claim. We can always find a gauge transformation such that $\mathbf{A}'$ satisfies

$$\nabla \cdot \mathbf{A}' = 0. \tag{8.4}$$

Making this choice is referred to as a *Coulomb Gauge*.

Proof. Suppose we have an $\mathbf{A}$ such that $\nabla \times \mathbf{A}$ gives us the magnetic field we want, but $\nabla \cdot \mathbf{A} = f(\mathbf{r})$, we instead choose the vector potential $\mathbf{A}' = \mathbf{A} + \nabla\chi$ so our divergence becomes

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \nabla \cdot \nabla\chi = f + \nabla^2\chi \tag{8.5}$$

Since we want the divergence of $\mathbf{A}'$ to be zero, we simply need to pick our function χ such that

$$\nabla^2\chi = -f \tag{8.6}$$

which is the familiar Poisson Equation, which we know always has a solution. □

We have seen that changes of coordinates and adding functional forms are types of gauge freedoms. You can imagine as theories get more complex, there are more complicated forms of gauge freedoms. However, these all stem from the fact that the laws of physics do not depend on the choices we make. Once we construct fields that are related to observables through derivatives, we supply extra degrees of freedom that do not affect the measurments of the observables.

Gauge Symmetry is really the expression of redundancy in mathematical expressions; there is not a unique correspondence between observations and the abstractions that desecribe them. If we wanted to, we could add as many functions as we wanted. There are many functions that belong to the same equivalence class of physically indistinguishable observations; members of this equivalence class are thereby related by gauge transformations.

9 The Cross-Product and Pseudovectors

You almost certainly have encountered the cross product in mechanics classes and highschool math classes.

Formally, the cross product is the bilinear map $T : V \times V \to V$ such that for vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$,

$$T(\mathbf{x}, \mathbf{y}) = \mathbf{x} \times \mathbf{y} = (x_2y_3 - x_3y_2)\hat{\imath} - (x_1y_3 - x_3y_1)\hat{\jmath} + (x_1y_2 - x_2y_1)\hat{k} \tag{9.1}$$

You may have taken on faith that this operation can only be done on 3-dimensional vector spaces. However, this is because there is an isomorphism (one-to-one correspondence) between the cross product and antisymmetric (or skew-symmetric) matrices. in dimension 3. We see that the only way so satisfy the equation

$$\frac{n(n-1)}{2} = n \tag{9.2}$$

Is if $n = 3$. Notice that the left-hand side is the dimension of an antisymmetric three by three matrix. That is, given any vector $\mathbf{x} = (x_1, x_2, x_3)$, we can define the bijection

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \tag{9.3}$$

The linear function represented by the matrix is

$$L(\mathbf{y}) = \mathbf{x} \times \mathbf{y} \tag{9.4}$$

Whenever we observe a cross product in electromagnetism, there is always an antisymmetric matrix somewhere, disguised as a vector. These tricksters—known as "pseudovectors"—can be recognized by the fact that their sign is just a matter of convention. It becomes immediately clear that the magnetic field vector $\mathbf{B}$ is an example of a pseudovector; we used the right-hand rule to point our thumb in the direction of the current, and curled or fingers in the direction of the magnetic field. We just as easily could have used our left hand, and nothing would change about the field except the sign.

In general, cross-products occur in pairs; one $\times$ to create a pseudovector, and another $\times$ to get rid of it.

Quantities of direct physical significance should not have an ambiguous sign. Therefore, pseudovectors are simply conveniences to aid in calculation. For instance, recalling the Lorentz Force Law in zero electric field,

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{9.5}$$

In this case, the magnetic field vector is a pseudovector; however, $\mathbf{B}$ itself may be calculated due to a cross product. The curl of a true vector $\mathbf{A}$ always yields a pseudovector; so we see that the equation becomes

$$\mathbf{F} = q\mathbf{v} \times (\nabla \times \mathbf{A}) \tag{9.6}$$

What this tells us is that the pseudovector $\mathbf{B}$ will act like a vector under proper rotation, but if we change a physical system to its mirror image, the pseudovector experiences a sign change. For instance, one can imagine a mirror world in which the left-hand-rule is used; the sign of a magnetic field would therefore be reversed.